Pages changed in the Enhanced Version

TI-Nspire CAS OS3 and Casio ClassPad version

CAMBRIDGE

ESSENTIAL
Further Mathematics

Fourth edition

ENHANCED

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New edition with CAS calculator updates

Interactive online textbook and PDF

Exercise 1A

1 What is:

a a numerical variable? Give an example. **b** a categorical variable? Give an example.

2 There are two types of numerical variables. Name them.

3 Classify each of the following variables as numerical or categorical. If the variable is numerical, further classify the variable as discrete or continuous.
Recording information on:

a length of bananas (in centimetres)
b number of cars in a supermarket car park
c daily temperature in °C
d eye colour (brown, blue, . . .)
e shoe size (6, 8, 10, . . .)
f the number of children in a family
g city of residence (NY, London, . . .)
h number of people who live in your city/area
i time spent watching TV (hours)
j the TV channel most watched by students
k salary (high, medium, low)
l salary (in dollars)
m whether a person smokes (yes, no)
n the number of cigarettes smoked per day

4 Classify the data for each of the variables in Table 1.1 as numerical or categorical.

1.2 Organising and displaying categorical data

Skillsheet

TEACHER

PowerPoint

The frequency table

With a large number of data values, it is difficult to identify any patterns or trends in the raw data. We first need to organise the data into a more manageable form. A statistical tool we use for this purpose is the **frequency table**.

The frequency table

A **frequency table** is a listing of the values a variable takes in a data set, along with how often (frequently) each value occurs.
Frequency can be recorded as a

- count: the number of times a value occurs, or
- per cent: the percentage of times a value occurs (**percentage frequency**)

$$\text{per cent} = \frac{\text{count}}{\text{total count}} \times 100\%$$

A listing of the values a variable takes, along with how frequently each of these values occurs in a data set, is called a **frequency distribution**.

Example 1 **Frequency table for a categorical variable**

The sex of 11 preschool children is as shown (F = female, M = male):

F M M F F M F F F M M

Construct a frequency table to display the data.

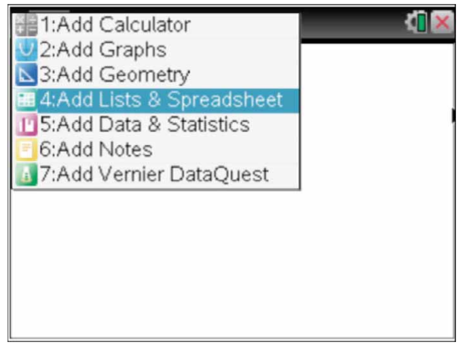
How to construct a histogram using the TI-Nspire CAS

Display the following set of marks in the form of a histogram.

16	11	4	25	15	7	14	13	14	12	15	13	16	14
15	12	18	22	17	18	23	15	13	17	18	22	23	

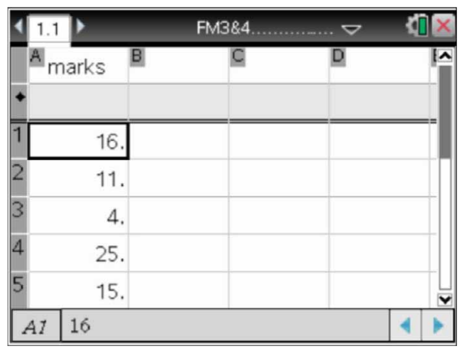
Steps

1 Start a new document: Press (on) and select **New Document** (or use (ctrl) + [N]). If prompted to save an existing document, move cursor to **No** and press (enter).



2 Select **Add Lists & Spreadsheet.** Enter the data into a list named *marks*.

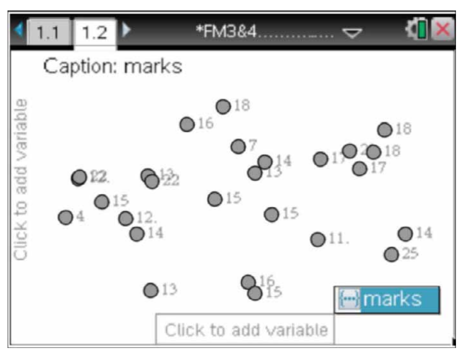
- **a** Move the cursor to the name space of column A (or any other column) and type in *marks* as the list name. Press (enter).
- **b** Move the cursor down to row 1, type in the first data value and press (enter). Continue until all the data has been entered. Press (enter) after each entry.



3 Statistical graphing is done through the **Data & Statistics** application. Press (ctrl) + [I] and select **Add Data & Statistics** (or press (on), arrow to, and press (enter)).

Note: A random display of dots will appear – this is to indicate that data are available for plotting. It is not a statistical plot.

- **a** Press (tab) to show the list of variables. The variable *marks* is shown as selected. Press (enter) to paste the variable *marks* to that axis.



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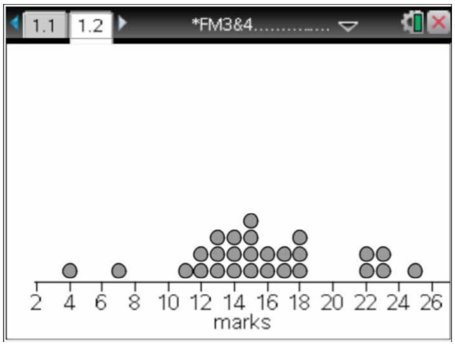
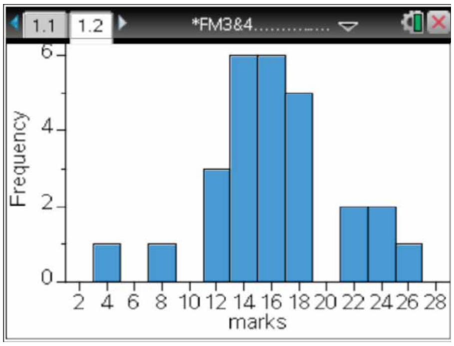
b A dot plot is then displayed as the default plot. To change the plot to a histogram press
(menu) **>Plot Type>Histogram**



Note for CX only: To add colour (or change colour) move cursor over the plot and press (ctrl) + (menu) **>Color>Fill Color**.

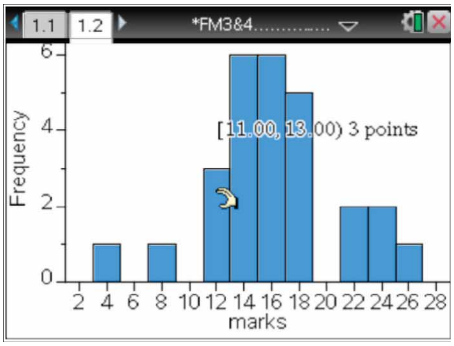
Your screen should now look like that shown opposite. This histogram has a column (or bin) width of 2 and a starting point of 3.



4 Data analysis

a Move cursor onto any column, will show and the column data will be displayed as shown opposite.

b To view other column data values move the cursor to another column.



Note: If you click on a column it will be selected. To deselect any previously selected columns, move the cursor to the open area and press .
Hint: If you accidentally move a column or data point, press (ctrl) + (esc) to undo the move.

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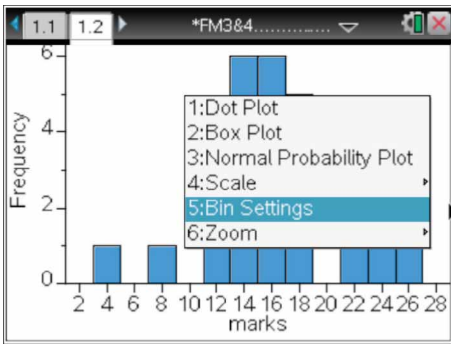
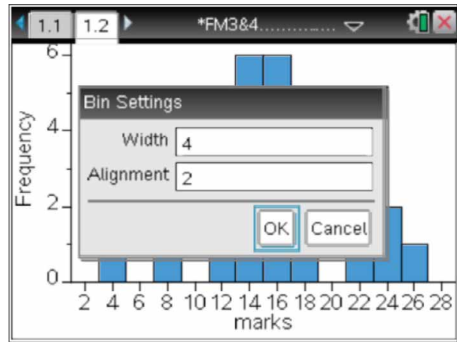
5 Change the histogram column (bin) width to **4** and the starting point to **2**.

a Press ctrl + menu to get the contextual menu as shown (below left).
Hint: Pressing ctrl + menu with the cursor on the histogram gives you access to a contextual menu that enables you to do things that relate only to histograms.

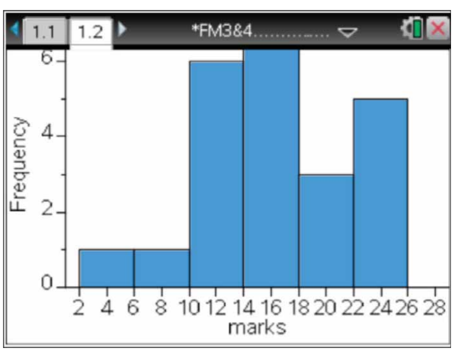
b Select **Bin Settings**.

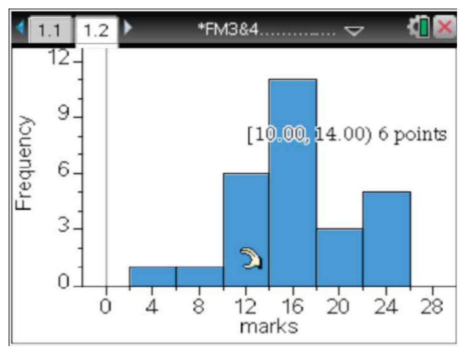
c In the settings menu (below right) change the **Width** to **4** and the **Starting Point** (**Alignment**) to **2** as shown. Press enter.



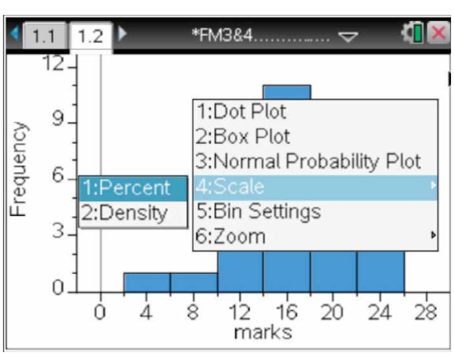


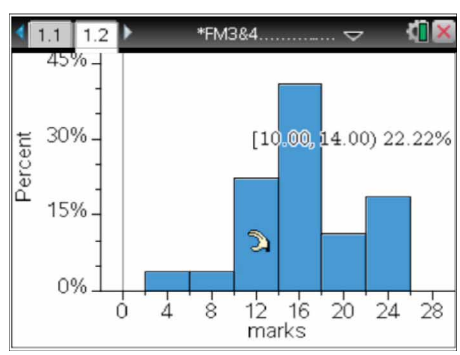
d A new histogram is displayed with a column width of 4 and a starting point of 2 but it no longer fits the viewing window (below left). To solve this problem press ctrl + menu **>Zoom>Zoom-Data** to obtain the histogram shown below right.





6 To change the frequency axis to a percentage axis, press ctrl + menu **>Scale>Percent** and then press enter.





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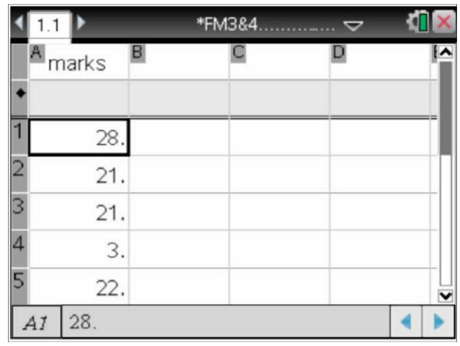
How to construct a box plot with outliers using the TI-Nspire CAS

Display the following set of 19 marks in the form of a box plot with outliers.

28 21 21 3 22 31 35 26 27 33 36 35 23 24
43 31 30 34 48

Steps

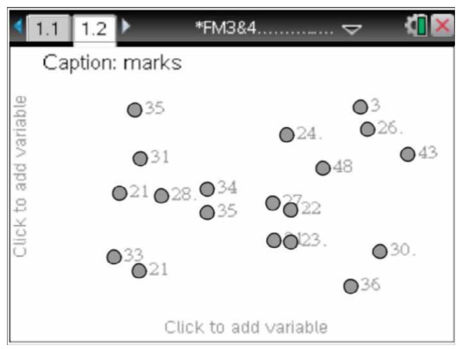
1. Start a new document: Press (on) and select **New Document** (or use (ctrl) + (N)).



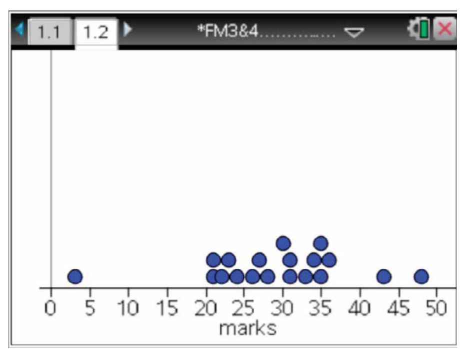
2. Select **Add Lists & Spreadsheet**.
   Enter the data into a list called *marks* as shown.
3. Statistical graphing is done through the **Data & Statistics** application.

   Press (ctrl)+(I) and select **Add Data & Statistics**
   (or press (on), arrow to the Data & Statistics icon, and press (enter)).

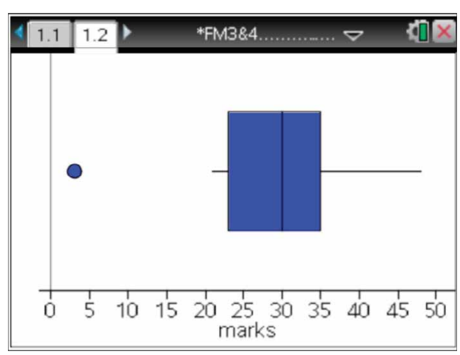
   Note: A random display of dots will appear – this is to indicate list data are available for plotting. It is not a statistical plot.



   a Press (tab) to show the list of variables. The variable *marks* is shown as selected. Press (enter) to paste the variable *marks* to that axis. A dot plot is displayed by default; see opposite.



   b To change the plot to a box plot press (menu) **>Plot Type>Box Plot**.
   Your screen should now look like that shown opposite.



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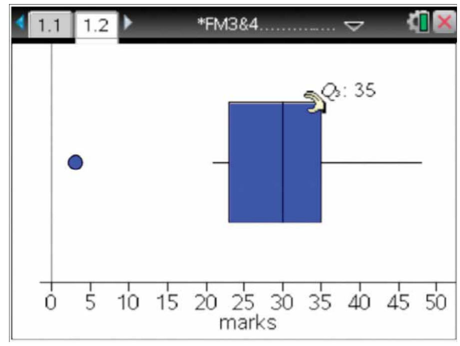
4 Data analysis

Key values can be read from the box plot by moving the cursor over the plot. (Run your finger or thumb gently over the touchpad to move the cursor.) On the Clickpad use the horizontal arrow keys (◀ and ▶) to move from point to point.



Starting at the far left of the plot, we see that the

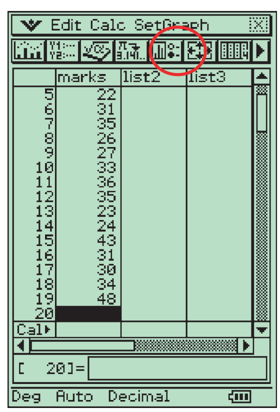
- minimum value is 3 (i.e. the outlier)
- lower adjacent value is 21
- first quartile is 23 (**Q_1 = 23**)
- median is 30 (**Median = 30**)
- third quartile is 35 (**Q_3 = 35**)
- maximum value is 48.

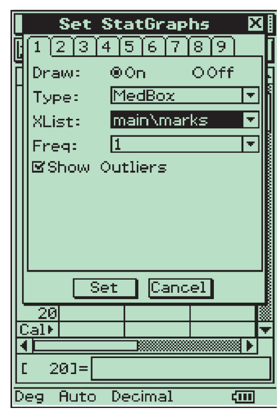
How to construct a box plot with outliers using the ClassPad

Display the following set of 19 marks in the form of a box plot with outliers.

28 21 21 3 22 31 35 26 27 33 36 35 23 24
43 31 30 34 48

1 Open the **Statistics** application and enter the data into the column labelled **marks**. Your screen should look like the one shown.





2 Open the **Set StatGraphs** dialog box by tapping [icon] in the toolbar. Complete the dialog box as given below.

- **Draw**: select **On**
- **Type**: select **MedBox** (▾)
- **XList**: select **main \ marks** (▾)
- **Freq**: leave as **1**

Tap the **Show Outliers** box to add a tick (☑).

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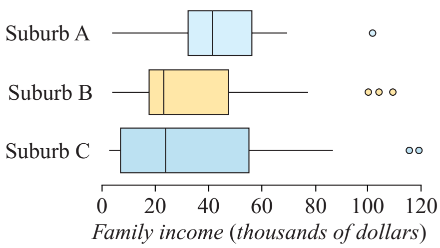
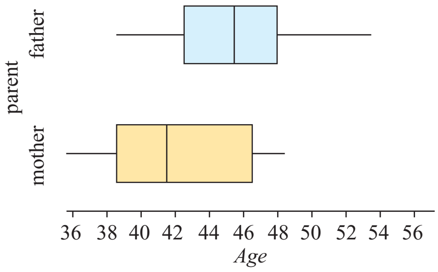
a On average, did the teacher tend to overestimate or underestimate her students' marks? Explain.
b Were the teacher's marks more or less variable than the actual marks? Explain.
c Compare the two distributions in terms of shape (including outliers, if any), centre and spread. Give appropriate values at a level of accuracy that can be read from the plot.
d Comment on how the predicted marks of the teacher compared to the students' actual marks.

6 A random sample of 250 families from three different suburbs was used in a study to try to identify factors that influenced a family's decision about taking out private health insurance. One variable investigated was family income. The information gathered on family incomes is presented opposite in the form of parallel box plots.



a In which suburb was the median household income the greatest?
b In which suburb were family incomes most variable?
c What do the outliers represent?
d Which of the following statements are true?
  i 'At least 75% of the families in Suburb A have an income that exceeds the median family income in Suburb B.'
  ii 'More than 50% of the families in Suburb A have incomes less than $45 000.'
  iii 'The distribution of family incomes in Suburb C is approximately symmetric.'
  iv 'The *mean* family income in Suburb B is greater than the *median* family income in Suburb B.'

7 The parallel box plots opposite display the distribution of age (in years) of the mothers and fathers of 26 students. Label each of the following statements as true or false.



a The median age of the mothers is less than the median age of the fathers.
b Approximately 75% of the fathers were 48 years old or younger.
c At least 75% of the mothers were younger than the median age of the fathers.
d Approximately 50% of the mothers were aged between 42 and 48 years.
e More than 25% of the fathers were aged 50 years or older.

For small data sets, it is reasonable to assume that almost all the data values lie within two standard deviations of the mean. Making this assumption,

the range ≈ four standard deviations (≈ means 'approximately equals')

$$\therefore \text{ one standard deviation} \approx \frac{\text{range}}{4} \quad \text{or} \quad s \approx \frac{R}{4}$$

A rule for estimating the standard deviation for small data sets

$$\text{standard deviation} \approx \frac{\text{range}}{4} \quad \text{or} \quad s \approx \frac{R}{4}$$

Example 2 Estimating the standard deviation

Estimate the value of the standard deviation of the data set 2 4 3 7 5 9 4 5 4 using the rule $s \approx \frac{R}{4}$.

Solution

1 Determine the value of the range, R. — $R = 9 - 2 = 7$

2 Substitute the value of R in the formula $s \approx \frac{R}{4}$. — $\therefore s \approx \frac{R}{4} = \frac{7}{4} = 1.75$

3 Write down your answer, in this case rounding to the nearest whole number. **Remember, you are only estimating**. — The estimated value of the standard deviation is 2.

Note: The true value is 2.1, correct to one decimal place.

Now that we have a way of checking the reasonableness of our results, we can feel confident about using a graphics calculator to calculate standard deviations.

How to calculate the mean and standard deviation using the TI-Nspire CAS

The following are the heights (in cm) of a group of women:

176 160 163 157 168 172 173 169

Determine the mean and standard deviation of the women's heights. Give your answers correct to two decimal places.

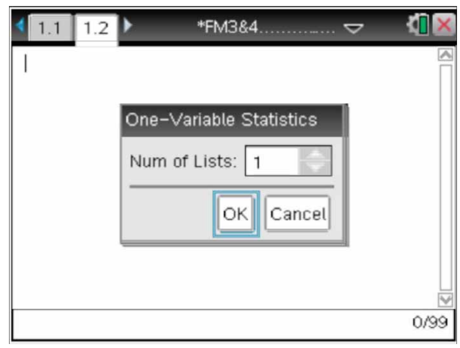
Steps

1 Start a new document by pressing (ctrl) + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *height*, as shown.

3 Statistical calculations can be done in either the **Lists & Spreadsheet** application or the **Calculator** application (used here).
Press (ctrl) + I and select **Add Calculator**.

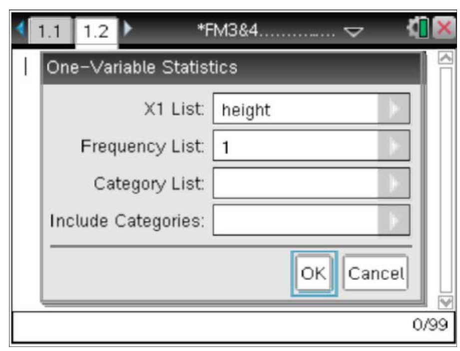
a Press (menu) **>Statistics>Stat Calculations>One-Variable Statistics**.
This will generate the pop-up screen shown opposite.



b As we only require the mean and standard deviation for one set of data

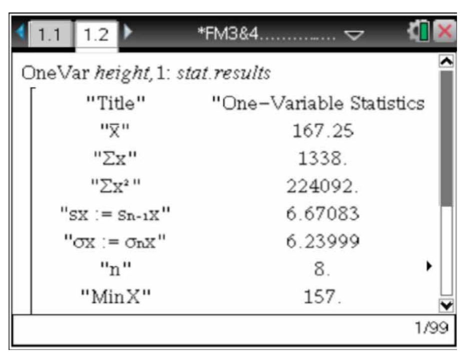
i Press (enter) to generate a second pop-up screen, as shown opposite.

ii To complete this screen, use the ▶ arrow and (enter) to paste in the list name *height*. Pressing (enter) exits this screen and generates the results screen shown next.



4 Write down the answers to the required degree of accuracy (i.e. two decimal places).

The mean height of the women is $\bar{x} = 167.25$ cm and the standard deviation is $s = 6.67$ cm.



Notes:

1 The sample standard deviation is **sx**.

2 Use the ▲▼ arrows to scroll through the results screen to obtain values for additional statistical values (i.e. Q_1, median, Q_3 and maximum value) if required.

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Selecting a sample

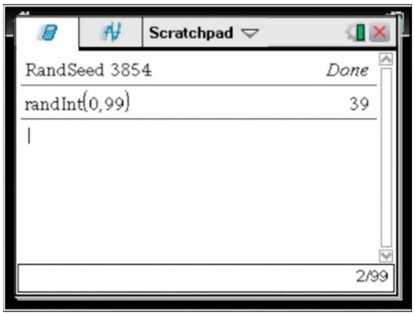
We could of course run the training program with all the students in the population of interest, but the training program is intensive so we decide to restrict the study to six students. The problem is to select at random six students from the population. One way of doing this is to write each person's ID number down on a piece of paper, put all the pieces of paper in a large container, thoroughly mix them up, and then draw out six numbers. The six students whose numbers have been chosen would then constitute an SRS chosen from a population made up of the 100 VCE students.

Another way is to use a graphics calculator to generate a set of six two-digit random numbers.

How to generate a sequence of random integers using the TI-Nspire CAS

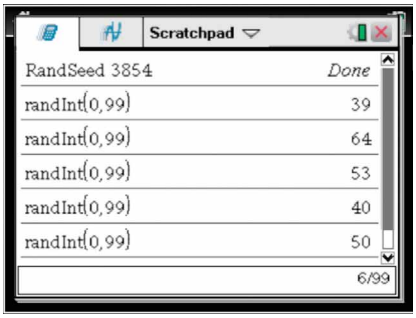
Generate a set of six random numbers between 00 and 99.

Steps



1 Press (⌂ on) (or (⌂) on the Clickpad), then [A] to open the **Scratchpad:Calculate**. Pressing (≡▦) also opens the **Scratchpad**. See Appendix for more details on the **Scratchpad**.

Note: You can also use **Documents>New Document>Add Calculator** if preferred.



2 **a** Set the seed: press (menu) **>Probability>Random>Seed** and type in any integer. (You could use the last four digits of your mobile number.) Press (enter).

b Use the **randInt( )** command to generate two-digit random integers between 0 and 99 (including 0 and 99).
Press: (menu) **>Probability>Random>Integer** and type **0,99** inside the brackets, as shown. Press (enter).
If your seed is different to the example shown, it is unlikely that your random integer will be the same as the one shown on the screen.

c Continue pressing (enter) to generate a sequence of two-digit random integers between 0 and 99.

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Review

Standard (z) scores

Standardised or **z-scores** are calculated by subtracting the mean from each data value and then dividing by the standard deviation.

The formula for calculating standard scores is $z = \dfrac{x - \bar{x}}{s}$

Example: A student obtains a mark of 76 in a subject where the mean mark is 60 and the standard deviation is 8. The standardised score is:

$$z = \frac{x - \bar{x}}{s} = \frac{76 - 60}{8} = \frac{16}{8} = 2$$

The value of the standard score gives the **distance** and **direction** of a data value from the **mean** in **standard deviations**.

For example, if a data value has a standardised score of:

- $z = 2.1$ the data value is **two** standard deviations **above** the mean.
- $z = 0$ the data value is **equal** to the mean.
- $z = -1$ the data value is **one** standard deviation **below** the mean.

In combination with the 68–95–99.7% rule, standard scores can be used to give a measure of the level of performance. For example, a student whose standardised score in a subject was

- $z = 2$ was in the top 2.5% of students in that subject
- $z = 0$ was exactly 'average' in that subject
- $z = -1.2$ was in the bottom 16% of students in that subject.

Simple random sample (SRS)

In a **simple random sample** each member of the population has an **equal chance** of being selected.

Skills check

Having completed this chapter you should be able to:

- calculate the mean and standard deviation of a data set
- estimate the size of the standard deviation of a data set using standard deviation $\approx \dfrac{\text{range}}{4}$, and use this estimate as a check when determining the standard deviation using a calculator
- understand the difference between the mean and the median as measures of centre and be able to identify situations where it is more appropriate to use the median
- know and be able to apply the 68–95–99.9% rule for bell-shaped distributions
- calculate standard or z-scores and use them to compare performance.

Review

12 In a normal distribution, approximately 16% of values lie:
A within one SD of the mean **B** within two SDs of the mean
C within three SDs of the mean **D** more than one SD above the mean
E more than two SDs below the mean

The following information relates to Questions 13 to 15
The ages of a group of 500 first-year university students are approximately normally distributed with a mean of 18.4 and a standard deviation of 0.3 years.

13 The percentage of students with ages between 17.8 and 19.0 years is:
A 5% **B** 16% **C** 50% **D** 68% **E** 95%

14 The number of students with ages more than 18.4 years is:
A 25 **B** 80 **C** 250 **D** 340 **E** 475

15 The number of students with ages less than 18.1 years is:
A 25 **B** 80 **C** 160 **D** 340 **E** 475

The following information relates to Questions 16 to 27.
Each week, a bus company makes 200 trips between two large country towns. The time taken for a bus to make the trip between the two towns is approximately normally distributed with a mean of 78 minutes and a standard deviation of 4 minutes.

16 The percentage of trips each week that take 78 minutes or more is:
A 16% **B** 34% **C** 50% **D** 68% **E** 84%

17 The percentage of trips each week that take between 74 and 82 minutes is:
A 16% **B** 34% **C** 50% **D** 68% **E** 95%

18 The percentage of trips each week that take less than 82 minutes is:
A 5% **B** 16% **C** 68% **D** 71.5% **E** 84%

19 The number of trips each week that take more than 70 minutes is:
A 5 **B** 97.5 **C** 100 **D** 190 **E** 195

20 The number of trips each week that take between 78 and 82 minutes is:
A 4 **B** 32 **C** 68 **D** 134 **E** 168

21 A trip that takes 86 minutes has a standardised time (z-score) of:
A -2 **B** -1 **C** 0 **D** 1 **E** 2

22 A trip that takes 71 minutes has a standardised time (z-score) of:
A -1.75 **B** -1.5 **C** -1.25 **D** 1.5 **E** 1.75

23 A trip that takes 89 minutes has a standardised time (z-score) of:
A 1.25 **B** 1.75 **C** 2.25 **D** 2.75 **E** 3.0

24 A standardised time for a trip is $z = 1.8$. The actual time (in minutes) is:
A 21.5 **B** 76.2 **C** 79.8 **D** 85.2 **E** 88.0

25 A standardised time for a trip is $z = -0.25$. The actual time (in minutes) is:

A 77 **B** 77.25 **C** 77.75 **D** 78.25 **E** 79

26 A standardised time for a trip is $z = -1.25$. The actual time (in minutes) is:

A 73 **B** 75 **C** 76.75 **D** 78.25 **E** 84

27 The time of a bus trip whose standardised time is $z = 2.1$ is:

A very much below average
B just below average
C around average
D just above average
E very much above average

28 The mean length of 10 garden stakes is $\bar{x} = 180.5$ cm. The standard deviation of the lengths is $s = 2.9$ cm. If the length of each garden stake is then reduced by exactly five centimetres, the mean and standard deviation of the lengths of the stakes will be:

A 175.5 cm and 2.4 cm
B 180.5 cm and 2.4 cm
C 175.5 cm and 2.9 cm
D 175.5 cm and 3.4 cm
E 185.5 cm and 2.9 cm

Extended-response questions

1 The stem plot opposite shows the distribution of urbanisation rates (percentage) for 23 countries.

a From the shape of the distribution, which measure of centre, the mean or the median, do you think would best indicate the typical urbanisation rate in these countries?

b Calculate both the mean and median and check your prediction.

Urbanisation

0	3 3 6 9 9 9
1	2 2 6 7
2	0 2 2 5 7 8 9
3	1 5
4	
5	4 6
7	
8	
9	9
10	0

2 a The lifetimes (in hours) of 15 batteries were measured with the following results:

30 34 31 39 58 31 36 34 61 37 31 44 43 35 65

What is a typical lifetime of the batteries measured? (Construct an appropriate stem plot to help you decide which measure of centre to use.)

Review

b The following data were collected in an investigation of the typical amount of soft drink dispensed by an automatic filling machine.

Fill number	1	2	3	4	5	6	7	8	9	10	11
Amount (millilitres)	204	206	194	210	198	204	200	198	205	200	199

From the data, what would you say is the typical amount of drink dispensed by the machine? (Construct an appropriate stem plot to help you decide which measure of centre to use.)

3 The foot lengths (in cm) of a random sample of 13 students are shown below:

30.9 32.1 31.8 30.7 31.9 29.4 31.6 33.3 30.7 31.6 30.8 31.2 32.2

a Estimate the standard deviation for the foot lengths.

b Calculate the mean and standard deviation of the foot lengths (to two decimal places).

c Determine the median foot length. Compare the median foot length with the mean foot length. What does this comparison tell you about the distribution of foot lengths?

4 Some IQ tests are set so that, on average, people taking the test score 100 points with a standard deviation of 15 points. IQ scores from this test are known to be approximately normally distributed.

a From this information we can conclude that:

i almost all people taking the test will score between ☐ and ☐

ii if you scored 90 points your score would be above/below average

iii if you scored between 85 and 115 you would be in the middle ☐ % of people taking the test

iv 50% of people taking the test will score more than ☐

v 99.85% of people taking the test will score more than ☐

vi 84% of people will score less than ☐

vii a score of 145 or above would place you in the top ☐ % of people taking this IQ test

viii 97.5% of people taking the test will score more than ☐

b **i** Nia's IQ score is 112. Determine Nia's standardised IQ score.

ii Jon's standardised IQ score is $z = -1.2$. Determine Jon's actual IQ score.

iii Given that 25% of people taking the test scored less than 90 and 75% of people scored less than 110, determine the *IQR*.

How to construct a scatterplot using the TI-Nspire CAS

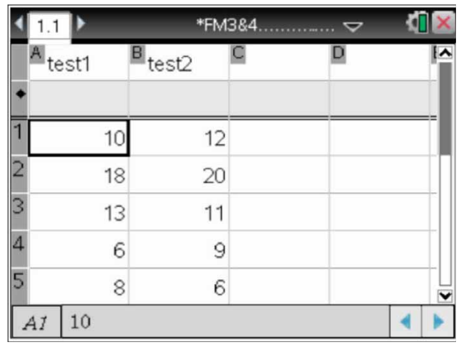
Construct a scatterplot for the set of test scores given below.
Treat Test 1 as the independent (i.e. x) variable.

Test 1	10	18	13	6	8	5	12	15	15
Test 2	12	20	11	9	6	6	12	13	17

Steps

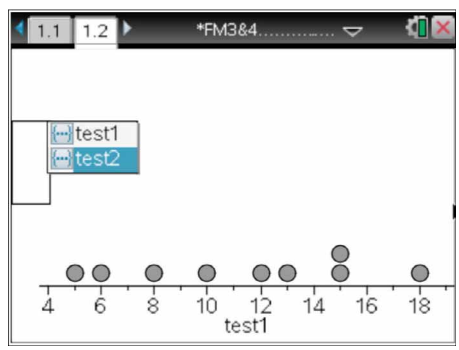
1 Start a new document by pressing ctrl + N (or ⌂ >**New Document**).

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *test1* and *test2*.



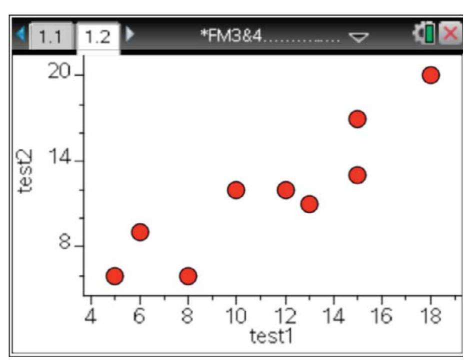
3 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + I and select **Add Data & Statistics**.
A random display of dots (not shown here) will appear – this is to indicate that list data is available for plotting. It is not a statistical plot.

a On this plot, press tab to show the list of variables. Select the independent variable, *test1*. Press enter to paste the variable to the x-axis.



b Press tab again and select the dependent variable, *test2*. Pressing enter pastes the variable to the y-axis and generates a scatterplot as shown below. The plot is scaled automatically.

Note: For CX only you can change the colour using ctrl + menu >**Color>Fill Color**.



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Determining the correlation coefficient using a graphics calculator

The graphics calculator automates the process of calculating a correlation coefficient. However, it does it as part of the process of fitting a straight line to the data (the topic of Chapter 5). As a result, more statistical information will be generated than you need at this stage.

How to calculate the correlation coefficient using the TI-Nspire CAS

Determine the value of the correlation coefficient, r, for the given data. Give the answer correct to two decimal places.

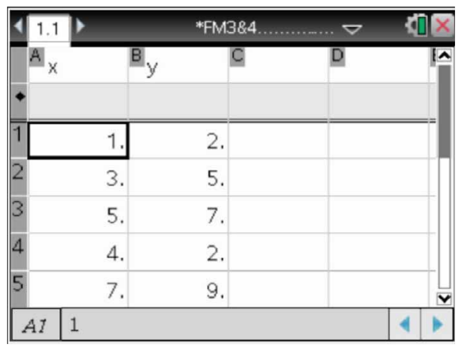
x	1	3	5	4	7
y	2	5	7	2	9

Steps

1 Start a new document by pressing ctrl + **N**.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named x and y.

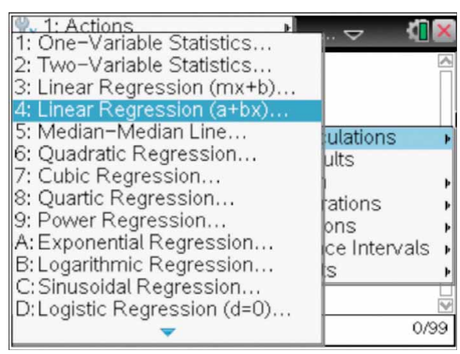
3 Statistical calculations can be done in the **Calculator** application (as used here) or the **Lists & Spreadsheet** application.
Press ctrl + **I** and select **Add Calculator**.



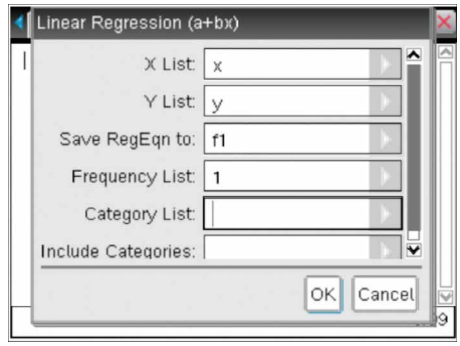
Method 1

Using the **Linear Regression (a+bx)** command:

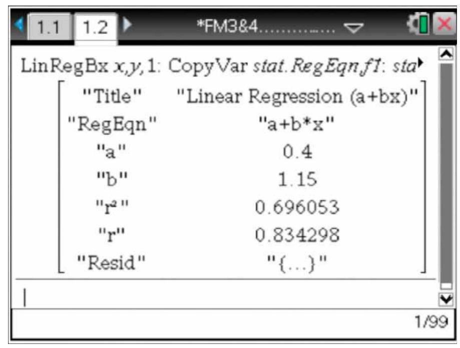
a Press menu **>Statistics>Stat Calculations>Linear Regression (a+bx)** to generate the screen opposite.



b Press (enter) to generate the pop-up screen, as shown. To select the variable for the X List entry, use ► and (enter) to select and paste in the list name *x*. Press (tab) to move to the Y List entry, use ► ▼ and (enter) to select and paste in the list name *y*.



c Press (enter) to exit the pop-up screen and generate the results shown in the screen opposite.

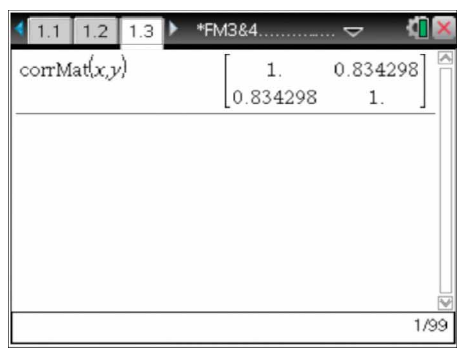


The value of the correlation coefficient is $r = 0.8342\ldots$ or 0.83, correct to two decimal places.

Method 2

Using the **corrMat(*x, y*)** command:
In the **Calculator** application, type in **corrmat(*x, y*)** and press (enter).
Alternatively:

a Press (catalog) (1) (C) to access the **Catalog**, scroll down to **corrMat(** and press (enter) to select and paste the **corrMat(** command onto the **Calculator** screen.



b Complete the command by typing in ***x, y*** and press (enter).

The value of the correlation coefficient is $r = 0.8342\ldots$ or 0.83, correct to two decimal places.

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a Name the dependent variable in the study.

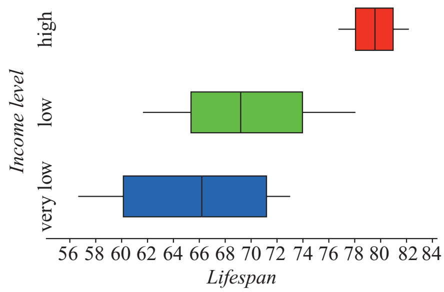
b Use a graphics calculator to construct a scatterplot of the data. Name variables *sleepdep* and *rem*.

c Does there appear to be a relationship between the variables? If so, is it positive or negative?

d Determine the value of r, the correlation coefficient, correct to three decimal places. Comment on the nature of the relationship between the variables in this study.

e Calculate the coefficient of determination (r^2) and interpret.

6 The parallel box plots below compare the distribution of average lifespan in a country (in years) for countries classified by average income level (very low, low and high).



a What is the median average life span (in years) for the:

i very low income countries?

ii high income countries?

b Complete the following sentences.

i Around ☐ % of low income countries have average life spans of less than 74 years.

ii ☐ % of high income countries have life spans greater than 76 years.

c The parallel box plots support the contention that average life span in a country is positively related to the median income level of a country. Explain why, giving the values of relevant medians as part of your explanation.

How to determine the equation of a least squares regression line using the formula

The heights (x) and weights (y) of 11 people have been recorded, and the values of the following statistics determined:

$\bar{x} = 173.2727$ cm $\quad s_x = 7.4443$ cm $\quad \bar{y} = 65.4545$ cm $\quad s_y = 7.5943$ cm $\quad r = 0.8502$

Use the formula to determine the equation of the least squares regression line that will enable weight to be predicted from height.

Steps

1 Identify and write down the IV and DV. Label as x and y, respectively.
Note: In saying that we want to predict weight from height, we are implying that height is the IV.

IV: height (x)
DV: weight (y)

2 Write down the given information.

$\bar{x} = 173.2727 \quad s_x = 7.4443$
$\bar{y} = 65.4545 \quad s_y = 7.5943$
$r = 0.8502$

3 Calculate the slope.

Slope:

$$b = \frac{rs_y}{s_x} = \frac{0.8502 \times 7.5943}{7.4443}$$
$$= 0.867 \text{ (correct to two d.p.)}$$

4 Calculate the intercept.

Intercept:

$$a = \bar{y} - b\bar{x}$$
$$= 65.4545 - 0.8673 \times 173.2727$$
$$= -84.8 \text{ (correct to one d.p.)}$$

5 Use the values of the intercept and the slope to write down the least squares regression line using the variable names.

$y = -84.8 + 0.867x$

or

weight $= -84.8 + 0.867 \times$ height

How to draw the graph and determine the equation of a least squares regression line using the TI-Nspire CAS

The following data give the heights (in cm) and weights (in kg) of 11 people.

Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height.

Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named *height* and *weight*, as shown.

3 Identify the independent variable (IV) and the dependent variable (DV).

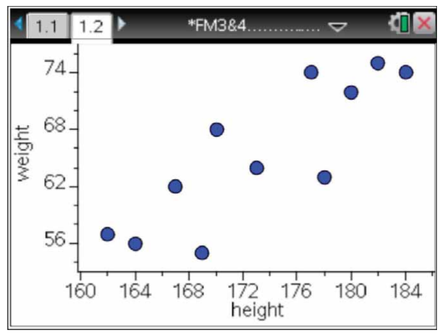
IV: height

DV: weight

Note: In saying that we want to predict *weight* from *height*, we are implying that *height* is the IV.

4 Press ctrl + I and select **Add Data & Statistics** and construct a scatterplot with the *height* (IV) on the horizontal (or x-) axis and *weight* (DV) on the vertical (or y-) axis.

If you need help to do this see page 105.



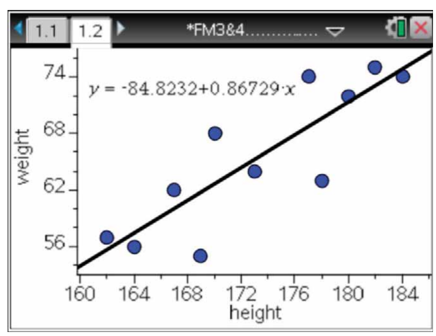
5 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the regression line on the scatterplot.

Note that, simultaneously, the equation of the regression line is shown.

The equation of the regression line is

$$y = -84.8 + 0.867x$$

or *weight* $= -84.8 + 0.867 \times$ *height*



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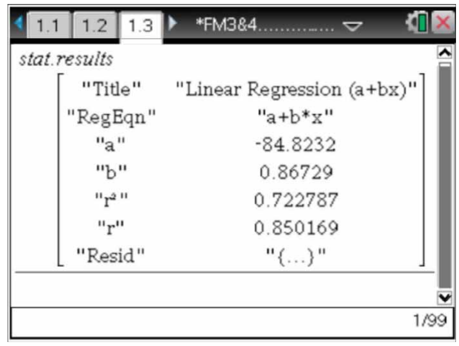
6 If you wish to have a full printout of the regression statistics:

a Press ctrl + I and select **Add Calculator** to open the **Calculator** application.

b Now press var and select **stat.results**. Press enter to display the full statistical information as shown opposite.



Note: Because you did a regression in the **Data & Statistics** application earlier, this information is stored as a **stat** variable and can be accessed using the var key.

7 Use the values of the intercept **a** and the slope **b** to write the equation of the least squares regression line using the variable names.

$weight = -84.8 + 0.867 \times height$

The coefficient of determination is $r^2 = 0.723$, correct to three decimal places.

How to draw the graph and determine the equation of a least squares regression line using the ClassPad

The following data give the heights (in cm) and weights (in kg) of 11 people.

Height (x)	177	182	167	178	173	184	162	169	164	170	180
Weight (y)	74	75	62	63	64	74	57	55	56	68	72

Determine and graph the equation of the least squares regression line that will enable weight to be predicted from height.

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Using a graphics calculator

In the regression analysis above, all statistical graphs and results were given. We will now show how they were generated with a graphics calculator.

How to conduct a regression analysis using the TINspire CAS

The data for this analysis is shown below.

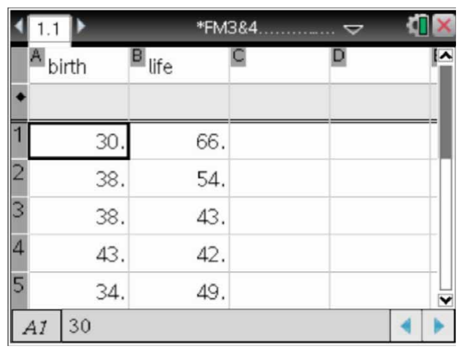
Birth rate (per thousand)	30	38	38	43	34	42	31	32	26	34
Life expectancy (years)	66	54	43	42	49	45	64	61	61	66

Steps

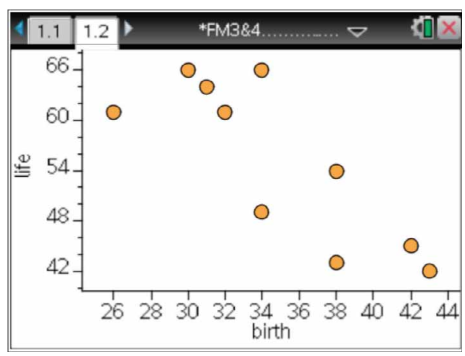
1 Write down the independent variable (IV) and dependent variable (DV). Use the abbreviations 'birth' for birth rate and 'life' for life expectancy.

IV: birth
DV: life

2 Enter the data into lists named *birth* and *life*, as shown.



3 Construct a scatterplot to investigate the nature of the relationship between life expectancy and birth rate.



4 Describe the relationship between life expectancy and birth rate as shown by the scatterplot. Mention direction, form, strength and outliers.

From the scatterplot we see that there is a moderately strong, negative, linear relationship between life expectancy and birth rate. There are no obvious outliers.

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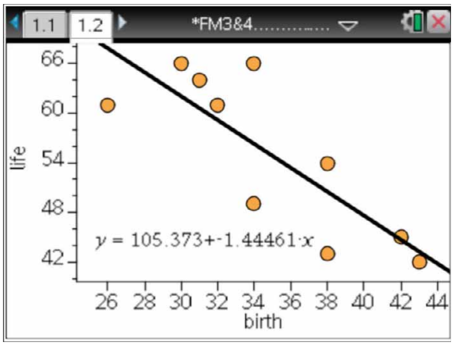
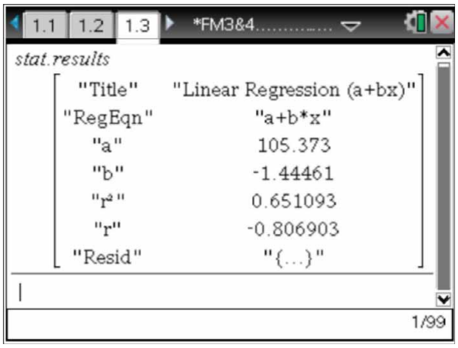
5 Find and plot the equation of the least squares regression line and generate the full list of regression statistics.



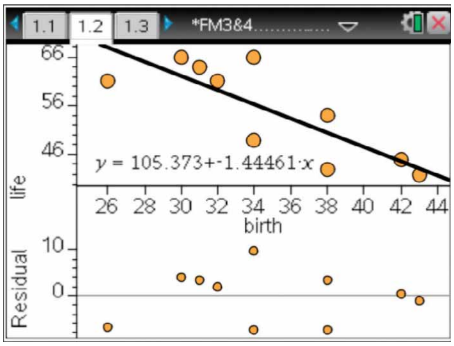


6 Generate a residual plot to test the linearity assumption.

Note: When you perform a regression analysis, the residuals are calculated automatically and stored as a list called ***stat.resid***.

Use ctrl + ◄ to return to the scatterplot.

Press menu **>Analyze>Residuals> Show Residual Plot** to display the residual plot on the same screen as the scatterplot.



7 Use the values of the intercept and slope to write the equation of the least squares regression line using the variable names. Also write the values of r and the coefficient of determination.

Regression equation:

$y = 105.4 - 1.445x$

or

life $= 105.4 - 1.445 \times$ birth

Correlation coefficient: $r = 0.8069$

Coefficient of determination: $r^2 = 0.651$

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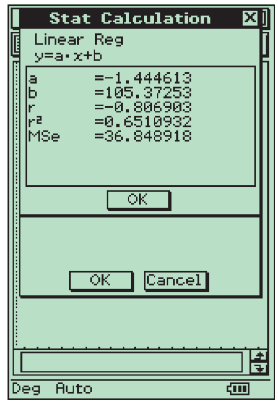
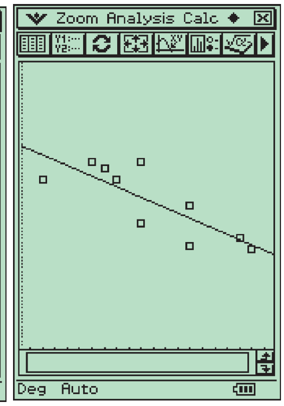
6 Tap **OK** to confirm your selections in the **Set Calculation** dialog box (above). This also generates the regression results.



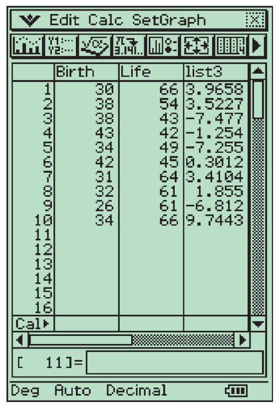
7 Tapping **OK** a second time automatically plots and displays the regression line on the scatterplot.

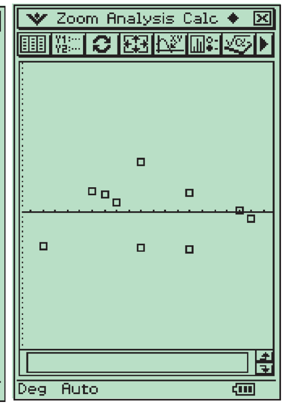
To obtain a full-screen plot, tap **Resize** from the icon panel.



8 Generate a residual plot to test the linearity assumption.

Note: When you performed a regression analysis earlier, the residuals were calculated automatically and stored in **list3**. The residual plot is a scatterplot with **list3** on the vertical axis and **birth** on the horizontal axis.





9 Use the values of the intercept and slope to write the equation of the least squares regression line using the variable names. Also write the values of r and the coefficient of determination.

Regression equation:

life = 105.4 − 1.445 × birth

Correlation coefficient: r = 0.8069

Coefficient of determination: $r^2 = 0.651$

Exercise 5C

1 The equation of a regression line that enables hand span to be predicted from height is:

$$\textit{Hand span} = 2.9 + 0.33 \times \textit{Height}$$

Complete the following sentences:

a The independent variable is ☐.

b The slope is ☐ and the intercept is ☐.

c A person is 160 cm tall. Using the regression equation, their predicted hand span is ☐ cm.

d If this person has an actual hand span of 58.5 cm, then the error of prediction (residual value) is ☐ cm.

2 The equation of a regression line that enables fuel consumption of a car (litres per 100 kilometres) to be predicted from its weight (kg) is:

$$\textit{Fuel consumption} = -0.1 + 0.01 \times \textit{Weight}$$

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Complete the following sentences:

a The dependent variable is $\boxed{}$.

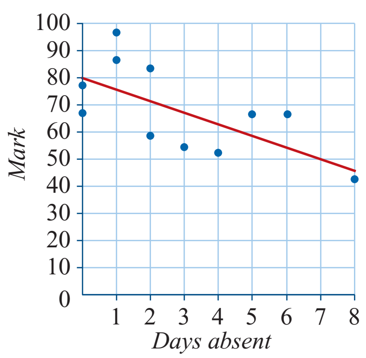
b The slope is $\boxed{}$ and the intercept is $\boxed{}$.

c A car weighs 980 kg and has an actual fuel consumption of 8.9 litres per 100 kilometres. Using this regression equation, the car's predicted fuel consumption is $\boxed{}$ litres per 100 kilometres.

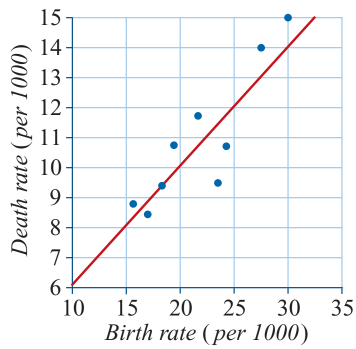
d The residual value for this car is $\boxed{}$ litres/100 kilometres.

3 Use the line on the graph to determine the equation of the regression line shown on each of the following scatterplots. Give the intercept correct to the nearest whole number and the slope correct to one decimal place.

a



b



4 The table below shows the scores obtained by nine students on two tests. We want to be able to predict Test B scores from Test A scores.

Test A score (x)	18	15	9	12	11	19	11	14	16
Test B score (y)	15	17	11	10	13	17	11	15	19

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *test a* and *test b*.

b Determine the equation of the least squares line along with the values of r and r^2.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

5 The table below shows the number of careless errors made on a test by nine students. Also given are their test scores. We want to be able to predict test score from the number of careless errors made.

Test score	18	15	9	12	11	19	11	14	16
Careless errors	0	2	5	6	4	1	8	3	1

Use your calculator to perform each of the following steps of a regression analysis.

a Construct a scatterplot. Name the variables *score* and *errors*.

b Determine the equation of the least squares line along with the values of r and r^2.

c Display the regression line on the scatterplot.

d Obtain a residual plot.

The squared transformation

Example 1 **Linearising the relationship with a squared transformation**

a Plot the data in the table, and comment on the form of the relationship between x and y.

x	0	1	2	3	4
y	2	3	6	11	18

b Apply a squared transformation to the x values (x^2), again plot the data, and comment on the form of the relationship between y and x^2.

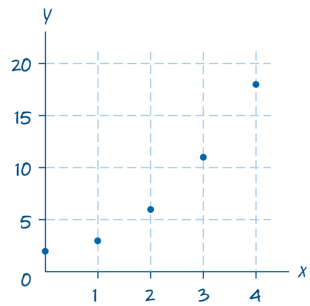
c Fit a line to the transformed data with y as the DV and x^2 as the IV. Write its equation.

d Use the equation determined from the transformed data to predict the value of y when $x = 5$.

Solution

a **1** Plot the values of y against x.

2 Decide if the form of the relationship is linear or non-linear.



3 Write down your conclusion.

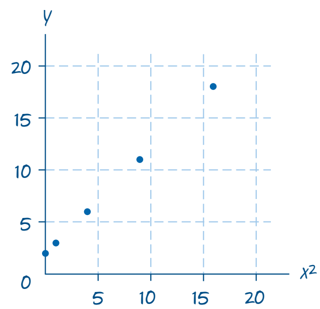
The relationship between y and x is non-linear.

b **1** Construct a new table of values.

x^2	0	1	4	9	16
y	2	3	6	11	18

2 Plot the values of y against x^2.

3 Decide if the form of the relationship is linear or non-linear.



4 Write down your conclusion.

The relationship between y and x^2 is linear.

c **1** Fit a straight line to the transformed data.

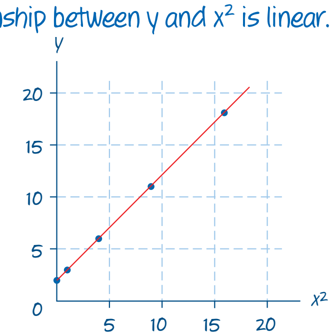
2 Write down the equation of the line noting that

i the y-intercept is 2 and the slope is 1

ii the independent variable on this graph is x^2 not x.

The equation of the line is:

$$y = 2 + x^2$$



d Use the non-linear equation $y = 2 + x^2$ to determine the value of y when $x = 5$.

When $x = 5$, $y = 2 + 5^2 = 27$

How to apply the squared transformation using the TI-Nspire CAS

Plot the data presented in the table below.

x	0	1	2	3	4
y	2	3	6	11	18

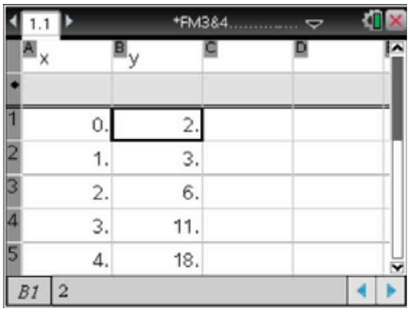
Use an x^2-transformation to linearise the data, then:

- fit a line to the transformed data with y as the DV and x^2 as the IV. Write its equation.
- use the equation determined from the transformed data to predict the value of y when $x = 5$.

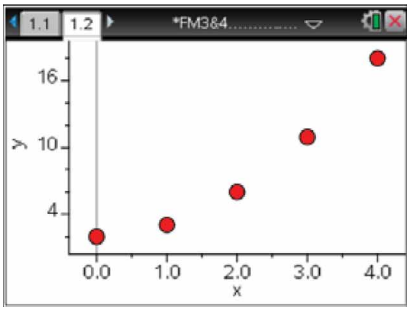
Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists named x and y, as shown.



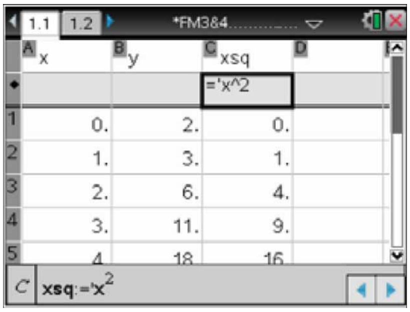
3 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of y against x. Let x be the independent variable and y the dependent variable. The plot is clearly non-linear.



4 Return to the **Lists & Spreadsheet** application (by pressing ctrl + ◀).
To calculate the values of x^2 and store them in a list named *xsq* (short for x-squared), do the following:

a Move the cursor to the top of column C and type ***xsq***. Press enter.

b Move the cursor to the grey cell immediately below the *xsq* heading. We need to enter the expression $= \mathbf{x}$^**2**. To do this, press = then **VAR** (var), highlight the variable **x** and then press enter to paste **x** into the formula line. Finally, type ^**2** (or press x^2) to complete the formula. Press enter to calculate and display the x-squared values.



Note: The dash in front of the x (i.e. $'x$) is automatically added when a list name is pasted from the **VAR** menu.

Note: You can also type in the variable **x** and then select **Variable Reference** when prompted. This avoids using the **VAR** menu.

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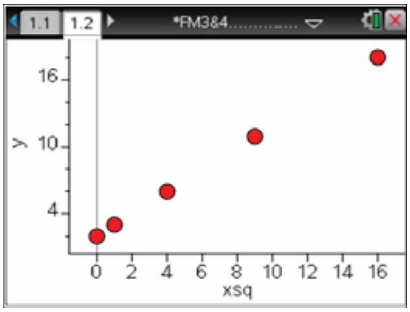
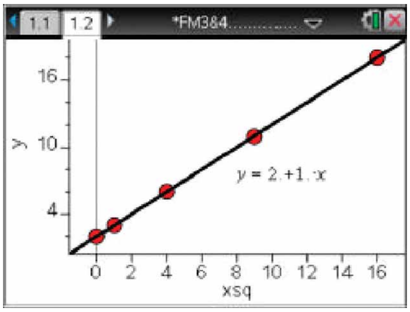
5 Construct a scatterplot of y against x^2.
Press ctrl + ▶ to return to the scatterplot created earlier and change the independent variable to *xsq* as follows:

a Press tab until the list of variables is displayed near the x-axis. Select the variable, *xsq*. Press enter to paste the variable to the x-axis.

b A scatterplot of y against *xsq* (x^2) is then displayed, as shown. The plot is clearly linear.



6 Press menu **>Analyze>Regression>Show Linear (a + bx)** to plot the regression line on the scatterplot.
Note that, simultaneously, the equation of the regression line is shown.
Note: The 'x on the screen corresponds to the transformed variable *xsq*, so we can rewrite the equations as $y = 2 + xsq$ or $y = 2 + x^2$.



7 Write down the equation for y in terms of x^2 and evaluate y when $x = 5$.

Eqn: $y = 2 + x^2$
when $x = 5$, $y = 2 + 5^2 = 27$

How to apply the squared transformation using the ClassPad

Plot the data presented in the table below.

x	0	1	2	3	4
y	2	3	6	11	18

Use an x^2-transformation to linearise the data, then:

- fit a line to the transformed data with y as the DV and x^2 as the IV. Write its equation.
- use the equation determined from the transformed data to predict the value of y when $x = 5$.

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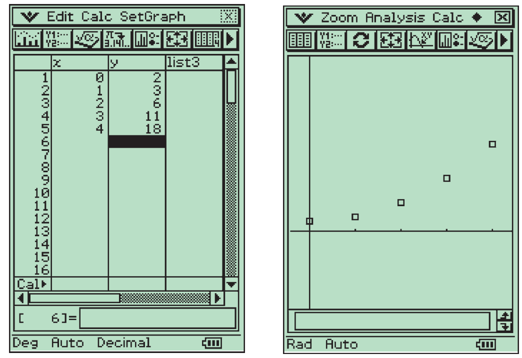
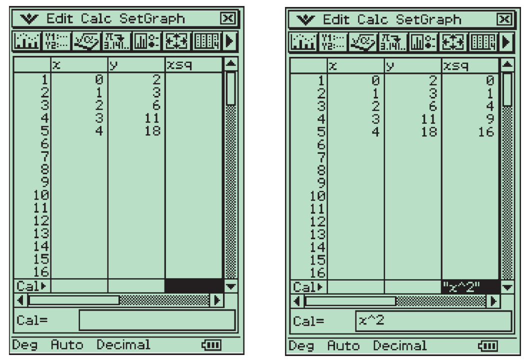
Steps

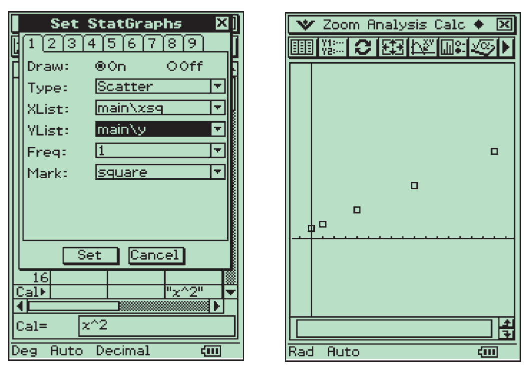
1. Open the **Statistics** application and enter the data into the columns named ***x*** and ***y***. Your screen should look like the one shown.
2. Construct a scatterplot of ***y*** against ***x***. Let x be the independent variable and y the dependent variable. The plot is clearly non-linear.



3. To calculate the values of x^2 and store them in a list named *xsq* (i.e. x-squared):
   - **a** Tap to highlight the cell at the top of the next empty list. Rename by typing ***xsq*** and pressing (EXE).
   - **b** Tap to highlight the cell at the bottom of the newly named *xsq* column (in the row titled **Cal ►**). Type **x^2** and press (EXE) to calculate and list the x^2 values.



4. Construct a scatterplot of y against *xsq* (i.e. x^2). The plot is clearly linear.

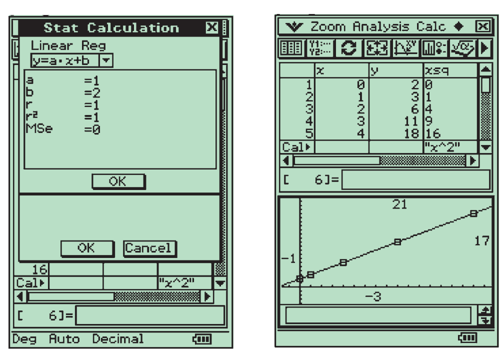


5. Fit a line to the transformed data and plot it on the scatterplot. Write down its equation using y as the DV and *xsq* (i.e. x^2) as the IV.

   $y = 2 + x^2$

   Use this equation to evaluate y when $x = 5$.

   When $x = 5$, $y = 2 + 5^2 = 27$



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Exercise 6B

The x squared transformation

These exercises are expected to be completed with the aid of a graphics calculator.

1 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	0	1	2	3	4
y	16	15	12	7	0

b Apply a squared transformation to the x values (x^2), again plot the data, and comment on the form of the relationship between y and x^2.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = -2$.

2 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	1	2	3	4	5
y	3	9	19	33	51

b Apply a squared transformation to the x values (x^2), again plot the data, and comment on the form of the relationship between y and x^2.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 6$.

3 a Plot the data in the following table, and comment on the form of the relationship between y and x.

x	1	2	3	4	5
y	30	27	22	15	6

b Apply a squared transformation to the x values (x^2), again plot the data, and comment on the form of the relationship between y and x^2.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 6$.

Data transformation is a process best performed totally within a graphics calculator environment and we will adopt this approach for the log x and $1/x$ transformations.

The log transformation

How to apply the log transformation using the TI-Nspire CAS

a Plot the data presented in the table below. Comment on the form of the relationship between x and y.

x	1	10	100	400	600	1000
y	0	10	20	25	28	30

b Apply a log transformation to the x values ($\log(x)$) and replot the data. Again comment on the relationship between x and y.

c Fit a line to the transformed data with y as the DV and log x as the IV. Write its equation.

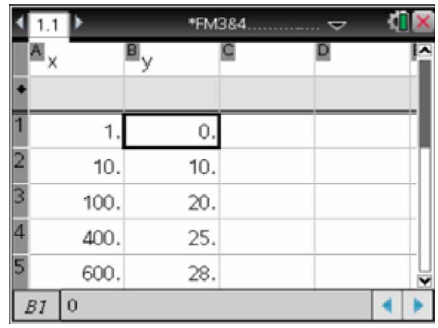
d Use the equation determined from the transformed data to predict the value of y when $x = 800$.

Steps

a **1** Start a new document by pressing ctrl + N.

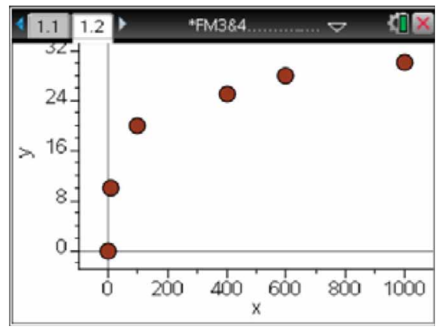
2 Select **Add Lists & Spreadsheet**.

Enter the data into lists named x and y, as shown opposite.



3 Press ctrl + I and select **Add Data & Statistics**.

Construct a scatterplot of y against x. Let x be the independent variable and y the dependent variable. The plot is clearly non-linear.

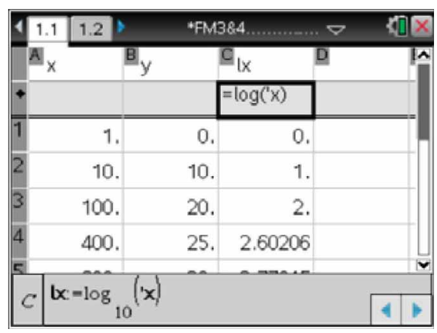


b Return to the **Lists & Spreadsheet** application (by pressing ctrl + ◀).

To calculate the values of log x and store them in a list named lx (short for log x), complete the following:

1 Move the cursor to the top of column C and type ***lx***. Press enter.

2 Move the cursor to the grey cell immediately below the lx heading and type $=$ **log(**. Then press **VAR** (var), highlight the variable **x**, press enter to paste **x** into the formula line, then type **)** to complete the command. Press enter to calculate and display the log values.

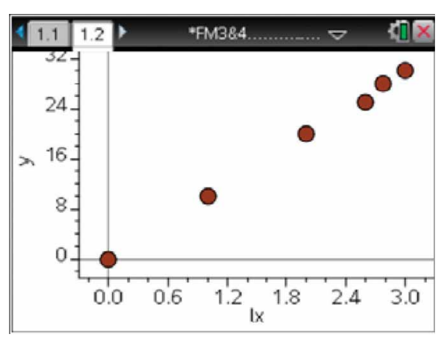


Note: If your answers are not given as decimals, refer to the Appendix to change **Mode** settings to **Approximate**.

3 Construct a scatterplot of y against log x.

Use ctrl + ▶ to return to the scatterplot created earlier and change the independent variable to lx.

A scatterplot of y against lx (i.e. the log of x) is displayed, as shown. The plot is clearly linear.



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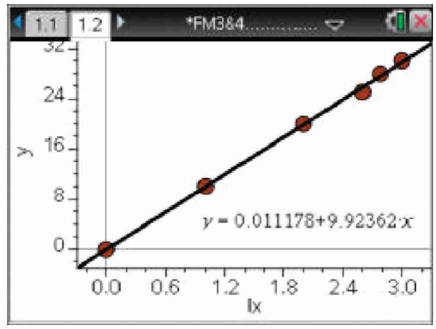
c Press (menu)**>Analyze>Regression>Show Linear (a + bx)** to plot the regression line on the scatterplot.
Note that, simultaneously, the equation of the regression line is shown.

Note: The 'x on the screen corresponds to the transformed variable lx (log x), so we can rewrite the equations as $y = 0.01 + 9.92\ lx$ or $y = 0.01 + 9.92 \log x$ (correct to two decimal places).



d Write down the equation for y in terms of $\log x$ and evaluate when $x = 800$.

Eqn: $y = 0.01 + 9.92 \log x$
When $x = 800$,
$y = 0.01 + 9.92 \log 800$
$= 28.8$ (to 1 d.p.)

How to apply the log transformation using the ClassPad

a Plot the data presented in the table below. Comment on the form of the relationship between x and y.

x	1	10	100	400	600	1000
y	0	10	20	25	28	30

b Apply a log transformation to the x values ($\log(x)$) and replot the data. Again comment on the relationship between x and y.

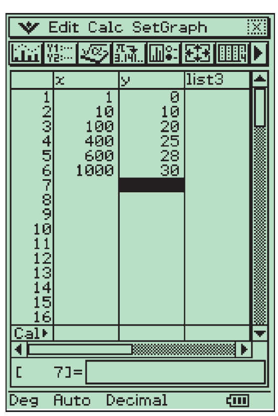
c Fit a line to the transformed data with y as the DV and $\log x$ as the IV. Write its equation.

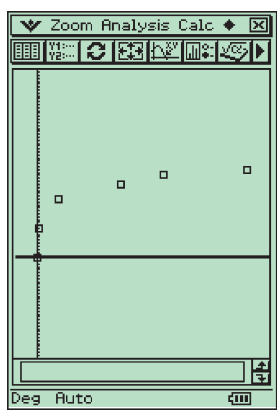
d Use the equation determined from the transformed data to predict the value of y when $x = 800$.

Steps

a 1 Open the **Statistics** application and enter the data into the columns named $\boldsymbol{x}$ and $\boldsymbol{y}$. Your screen should look like the one shown opposite.

2 Construct a scatterplot of $\boldsymbol{y}$ against $\boldsymbol{x}$. Let x be the independent variable and y the dependent variable. The plot is clearly non-linear.





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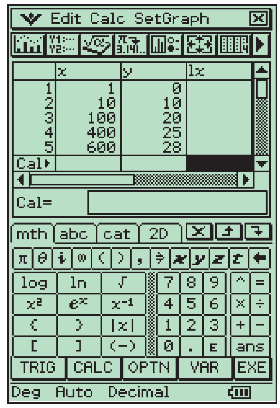
b To calculate the values of log x and store them in a list named ***lx*** (short for log x):

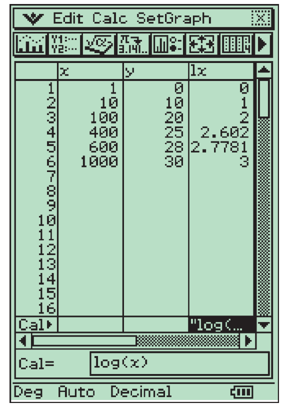
1 Tap to highlight the cell at the top of the next empty list (in this case, **list3**). Rename by typing ***lx*** and pressing (EXE).

2 Tap to highlight the cell at the bottom of the newly named ***lx*** column (in the row titled **Cal ►**).

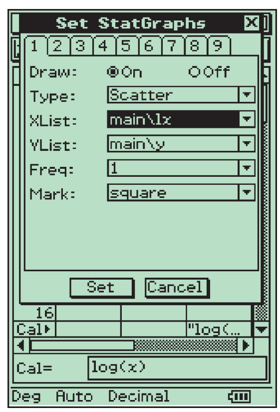
Typing **log(*x*)** and pressing (EXE) calculates and lists the values of log x.

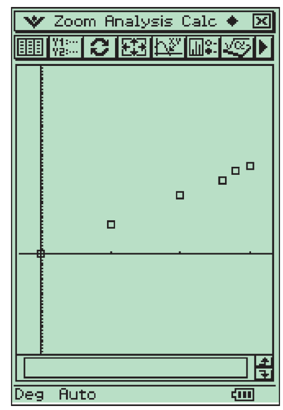
3 Construct a scatterplot of ***y*** against ***lx*** (i.e. log x). The plot is clearly linear.





Note: To ensure decimal values are displayed, **Decimal** should be visible in the status bar (at the bottom). If **Standard** is visible, tap **Standard** and it will change to **Decimal**.





c Fit a line to the transformed data and plot it on the scatterplot. Write down its equation using y as the DV and lx (log x) as the IV.

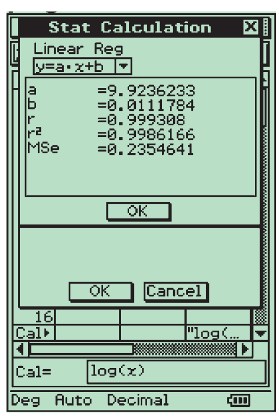
$y = 0.01 + 9.92 \log x$

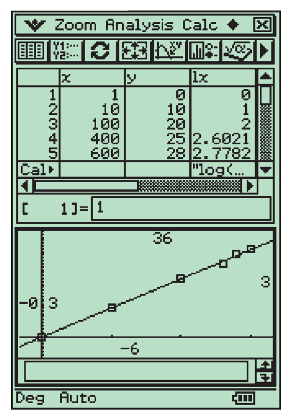
d Use this equation to evaluate y when x = 800.

When $x = 800$,

$y = 0.01 + 9.92 \log 800$

$= 28.8$ (to 1d.p.).





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Exercise 6C

The log x transformation

1 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	1	10	100	400	600	1000
y	30	20	10	5	2	0

b Apply a log transformation to the x values ($\log x$), again plot the data, and comment on the form of the relationship between y and $\log x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 500$. Give your answer correct to one decimal place.

2 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	5	10	150	500	1000
y	3.1	4.0	7.5	9.1	10.0

b Apply a log transformation to the x values ($\log x$), again plot the data, and comment on the form of the relationship between y and $\log x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 100$.

3 a Plot the data in the following table, and comment on the form of the relationship between y and x.

x	10	44	132	436	981
y	15.0	11.8	9.4	6.8	5.0

b Apply a log transformation to the x values ($\log x$), again plot the data, and comment on the form of the relationship between y and $\log x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 1000$.

The reciprocal $(1/x)$ transformation

How to apply the reciprocal transformation $\left(\frac{1}{x}\right)$ using the TI-Nspire CAS

a Plot the data presented in the table below. Comment on the form of the relationship between x and y.

x	1	2	3	4	5
y	30	15	10	7.5	6

b Apply a reciprocal transformation to the x values $(1/x)$ and replot the data. Again comment on the form of the relationship between x and y.

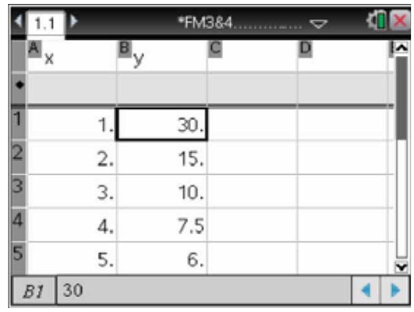
c Fit a line to the transformed data with y as the DV and $1/x$ as the IV. Write its equation.

d Use the equation determined from the transformed data to predict the value of y when $x = 4$.

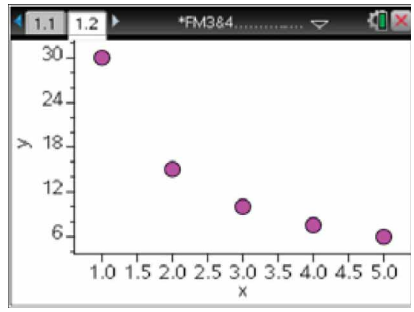
Steps

a **1** Start a new document by pressing (ctrl) + [N].

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named x and y, as shown opposite.



3 Press (ctrl) + [I] and select **Add Data & Statistics**.
Construct a scatterplot of y against x. Let x be the independent variable and y the dependent
variable. The plot is clearly non-linear.



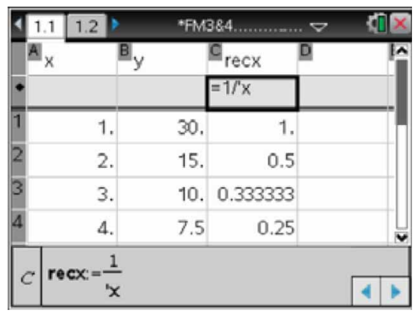
b Return to the **Lists & Spreadsheet** application (by pressing (ctrl) + ◀).

To calculate the values of $\frac{1}{x}$, complete the following:

1 Move the cursor to the top of column C and type ***recx*** (short for the reciprocal of x). Press (enter).

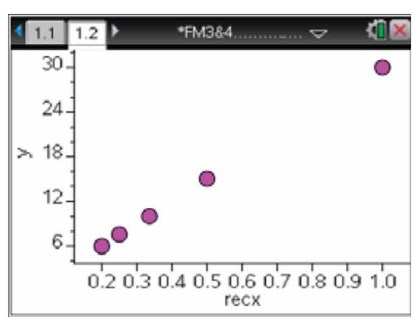
2 Move the cursor to the grey cell immediately below the *recx* heading and type $= 1 \div$, then press **VAR** ((var)) and highlight the variable **x** and press (enter) to paste into the formula

line. Press (enter) to calculate and display the $1/x$ values.



Note: If your answers are not presented as decimals, refer to the Appendix to change **Mode** settings to **Approximate**.

3 Construct a scatterplot of y against $1/x$ (i.e. *recx*)
Use (ctrl) + ▶ to return to the scatterplot created earlier and change the independent variable to *recx*.
A scatterplot of y against *recx* (the reciprocal of x) is displayed as shown.
The plot is clearly linear.



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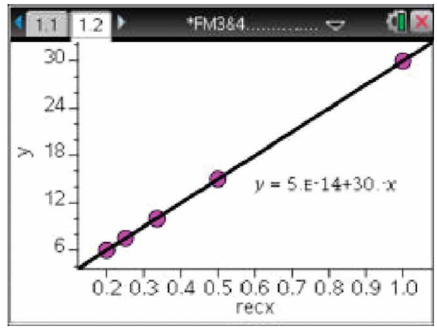
c Press (menu)**>Analyze>Regression>Show Linear (a + bx)** to plot the regression line on the scatterplot. Note that, simultaneously, the equation of the regression line is shown.



Note: The 'x on the screen corresponds to the transformed variable *recx* $(1/x)$, so we can rewrite the equations as $y = 5.E\text{-}14 + 30$ *recx* or $y = 30/x$ (we can ignore 5.E-14 because $5.E\text{-}14 = 5 \times 10^{-14} \approx 0$).

d Write down the equation for y in terms of $1/x$ and evaluate y when $x = 4$

Eqn: $y = 30/x$
When $x = 4$, $y = 30/4 = 7.5$

How to apply the reciprocal transformation using the ClassPad

a Plot the data presented in the table below. Comment on the relationship between x and y.

x	1	2	3	4	5
y	30	15	10	7.5	6

b Apply a reciprocal transformation to the x values $(1/x)$ and replot the data. Again comment on the relationship between x and y.

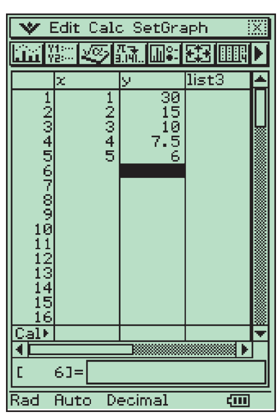
c Fit a line to the transformed data with y as the DV and $1/x$ as the IV. Write its equation.

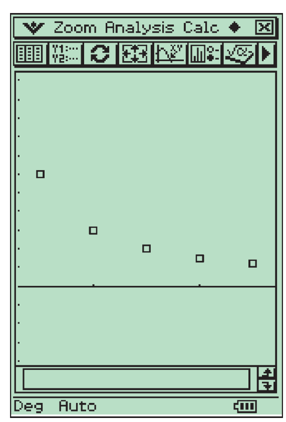
d Use the equation determined from the transformed data to predict the value of y when $x = 4$.

Steps

a **1** Open the **Statistics** application and enter the data into the columns named ***x*** and ***y***. Your screen should look like the one shown opposite.

2 Construct a scatterplot of ***y*** against ***x***. Let x be the independent variable and y the dependent variable. The plot is clearly non-linear.





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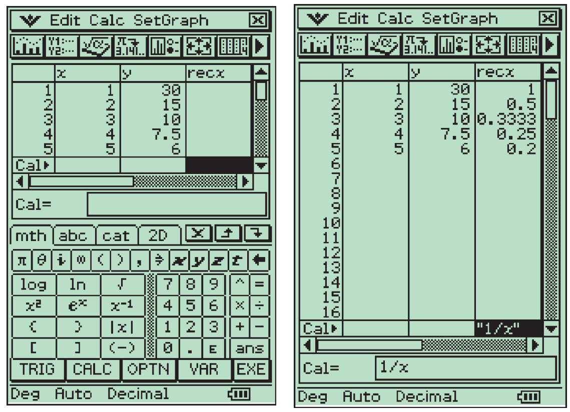
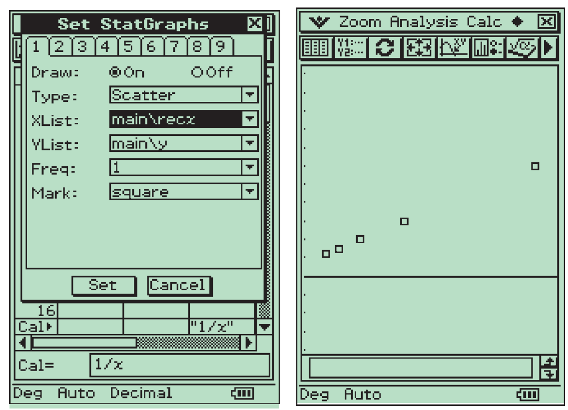
b To calculate the values of $1/x$ and store them in a list named *recx* (short for the reciprocal of x):

1. Tap to highlight the cell at the top of the next empty list (in this case, **list3**). Rename by typing ***recx*** and pressing (EXE).
2. Tap to highlight the cell at the bottom of the newly named **recx** column (in the row titled **Cal ▶**). Typing **1** ÷ ***x*** and pressing (EXE) calculates and lists the $1/x$ values.
3. Construct a scatterplot of ***y*** against ***recx*** (i.e. $1/x$). The plot is clearly linear.



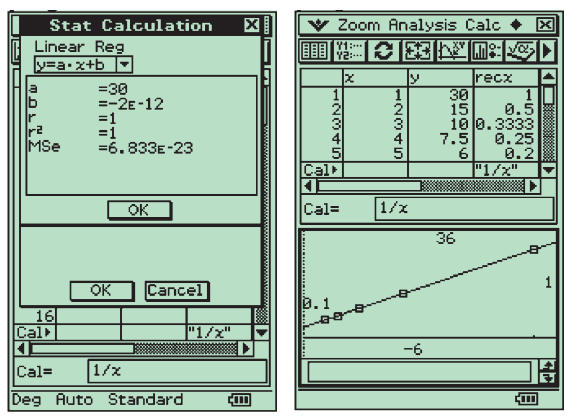


c Fit a line to the transformed data and plot it on the scatterplot. Write down its equation using y as the DV and *recx* $(1/x)$ as the IV.

$y = 30/x$

d Use this equation to evaluate y when $x = 4$.

When $x = 4$, $y = 30/4 = 7.5$



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Exercise 6D

The reciprocal (1/x) transformation

1 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	2	4	6	8	10
y	60	30	20	15	12

b Apply a reciprocal transformation to the x values $(1/x)$, again plot the data, and comment on the form of the relationship between y and $1/x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 5$.

2 a Plot the data in the table, and comment on the form of the relationship between y and x.

x	1	2	3	4	5
y	61	31	21	16	13

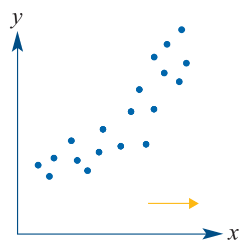
b Apply a reciprocal transformation to the x values $(1/x)$, again plot the data, and comment on the form of the relationship between y and $1/x$.

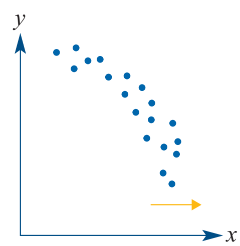
c Fit a line to the transformed data and write down its equation. Use the equation to predict the value of y when $x = 6$.

Which transformation?

What sorts of non-linear relationships can we linearise using the x^2 transformation?

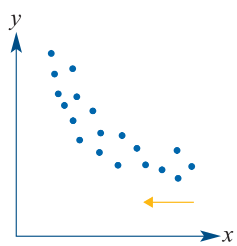
The x^2 transformation has the effect of stretching out the upper end of the x scale. As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the x to x^2 transformation. Note that for the x^2 transformation to apply, the scatterplot should peak or bottom around $x = 0$.

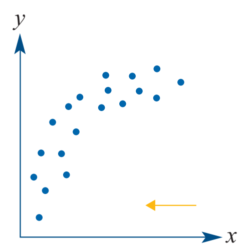




What sorts of non-linear relationships can we linearise using the log x transformation?

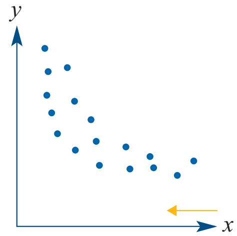
The log x transformation has the effect of compressing the upper end of the x scale. As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the x to log x transformation.

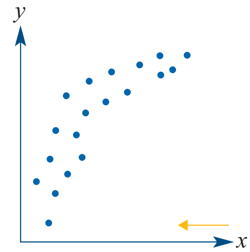




What sorts of non-linear relationships can we linearise using the $\frac{1}{x}$ transformation?

As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the x to $\frac{1}{x}$ transformation.





Exercise 6E

1 The table below shows the diameter (in m) of a number of umbrellas, along with the number of people each umbrella is designed to keep dry.

Diameter	0.50	0.70	0.85	1.00	1.10
Number of people	1	2	3	4	5

a Construct a scatterplot showing the relationship between *number of people* and umbrella *diameter*, and comment on the form.

b Apply a squared transformation to the x values (x^2), again plot the data, and comment on the form of the relationship between y and x^2.

c Fit a line to the transformed data and write down its equation. Give coefficients correct to one decimal place. Use the equation to predict the number of people that can be kept dry with an umbrella of diameter 1.30 m. Give answer correct to the nearest whole number.

2 The table shows the horsepower and fuel consumption in kilometres/litre of several cars.

Fuel consumption	5.2	7.3	12.6	7.1	6.3	10.1	10.5	14.6	10.9	7.7
Horsepower	155	125	75	110	138	88	80	70	100	103

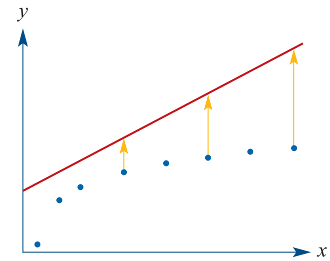
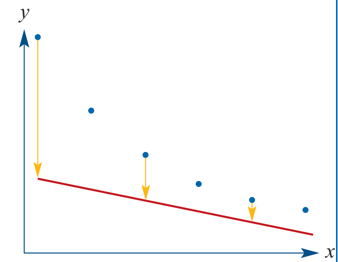
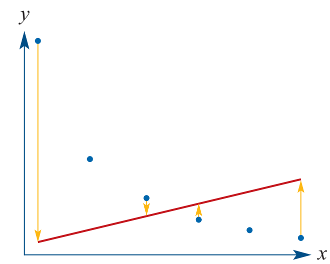
a Construct a scatterplot showing the relationship between *horsepower* and *fuel consumption*, and comment on the form. The IV is fuel consumption.

b Apply a reciprocal transformation to the x values ($1/x$), again plot the data and comment on the form of the relationship between y and $1/x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the horsepower of a car with fuel consumption of 9 kilometres per litre.

6.3 Transforming the *y*-axis

Another way to linearise the relationship between x and y is to apply these transformations to the y-axis. *Transforming the y-axis* will have the effect of moving the y values on the plot vertically, and leave the x values unaltered. The square, log and reciprocal transformations can be applied to the y-axis with the following effects:

Transformation	Outcome	Graph
y^2	Spreads out the large y values relative to the smaller data values	
$\log y$	Compresses large y values relative to the smaller data values	
$\frac{1}{y}$	Also compresses large y values relative to the smaller data values, to a greater extent than $\log y$. Note that values of y less than 1 become greater than 1, and values of y greater than 1 become less than 1, so that the order of the data values is reversed.	

Example 2 Linearising the relationship with a y-squared transformation

a Plot the data in this table, and comment on the form of the relationship between y and x.

x	0	1	2	3	4	5
y	0	3.2	4.5	5.5	6.3	7.1

b Apply a squared transformation to the y values (y^2), again plot the data, and comment on the form of the relationship between y^2 and x.

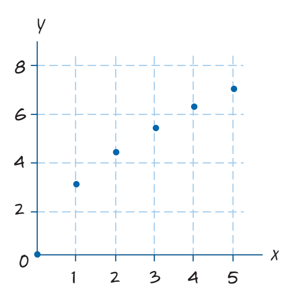
c Fit a line to the transformed data with y^2 as the DV and x as the IV. Write its equation and determine the value of y when $x = 1.6$.

Solution

a **1** Plot the values of y against x.

2 Decide if the form of the relationship is linear or non-linear.

The relationship between y and x is non-linear.



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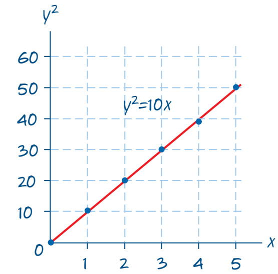
b **1** Construct a new table of values.

x	0	1	2	3	4	5
y^2	0	10.2	20.3	30.3	39.7	50.4

2 Plot the values of y^2 against x.

3 Decide if the form of the relationship is linear or non-linear.

The relationship between y^2 and x is linear, so that we can fit a line to the data.



c The slope of the line is approximately 10 and the intercept is approximately zero. Thus, remembering that the DV is y^2 and the IV is x, the equation of the line is $y^2 = 10x$.

The equation $y^2 = 10x$.
When $x = 1.6$,
$y^2 = 10 \times 1.6 = 16$ or $y = \pm 4$.

Example 3 Linearising the relationship with the log transformation

x	0	1	2	3	4	5
y	100	37	14	5	2	1

a Plot the data in this table, and comment on the form of the relationship between y and x.

b Apply a log transformation to the y values ($\log y$), again plot the data, and comment on the form of the relationship between $\log y$ and x.

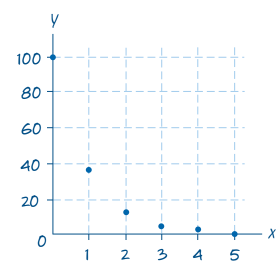
c Fit a line to the transformed data with $\log y$ as the DV and x as the IV. Write its equation and find the value of $\log y$ when $x = 1.5$.

Solution

a **1** Plot the values of $\log y$ against x.

2 Decide if the form of the relationship is linear or non-linear.

The relationship between y and x is non-linear.

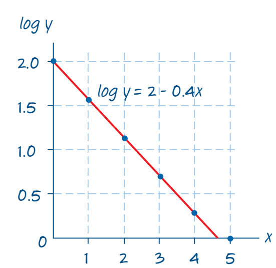


b **1** Construct a new table of values.

x	0	1	2	3	4	5
log y	2.00	1.57	1.15	0.70	0.30	0.00

2 Plot the values of $\log y$ against x.

3 Decide if the form of the relationship is linear or non-linear. The relationship between y and x is linear, so that we can fit a line to the data.



c The slope of the line is approximately −0.4 and the intercept is approximately 2. Thus, remembering that the DV is log y and the IV is x, the equation of the line is log $y = 2 - 0.4x$.

The equation is $\log y = 2 - 0.4x$.

When $x = 1.5$, $\log y = 2 - 0.4 \times 1.5 = 1.4$

or $y = 10^{1.4} = 25$.

Example 4 Linearising the relationship with the 1/y transformation

a Plot the data in this table, and comment on the form of the relationship between y and x.

x	1	2	3	4	5
y	10.0	5.0	3.3	2.5	2.0

b Apply a reciprocal transformation to the y values ($1/y$), again plot the data, and comment on the form of the relationship between x and $1/y$.

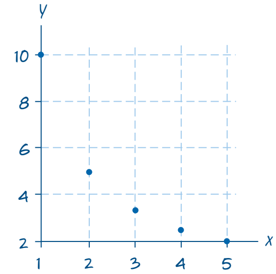
c Fit a line to the transformed data with log y as the DV and x as the IV. Write its equation and find the value of y when $x = 8$.

Solution

a **1** Plot the values of y against x.

2 Decide if the form of the relationship is linear or non-linear.

The relationship between y and x is non-linear.



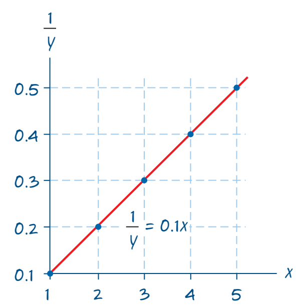
b **1** Construct a new table of values.

x	1	2	3	4	5
1/y	0.1	0.2	0.3	0.4	0.5

2 Plot the values of $\frac{1}{y}$ against x.

3 Decide if the form of the relationship is linear or non-linear.

The relationship between $\frac{1}{y}$ and x is linear, so that we can fit a line to the data.



The relationship between $\frac{1}{y}$ and x is linear.

c Thus, remembering that the DV is $\frac{1}{y}$ and the IV is x, the equation of the line is $\frac{1}{y} = 0.1x$.

The equation is $\frac{1}{y} = 0.1x$.

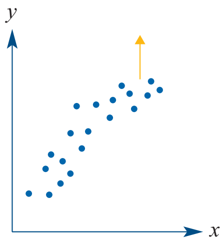
When $x = 8$,

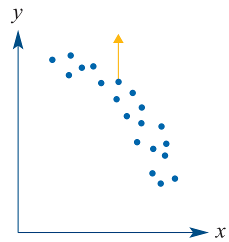
$\frac{1}{y} = 0.1 \times 8 = 0.8$ or $y = 1.25$.

Which transformation?

What sorts of non-linear relationships can we linearise using the y^2 transformation?

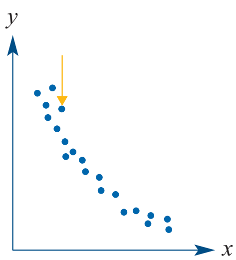
The y^2 transformation has the effect of stretching out the upper end of the y scale. As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the y to y^2 transformation. Note that for the y^2 transformation to apply, the scatterplot should peak or bottom around $y = 0$.

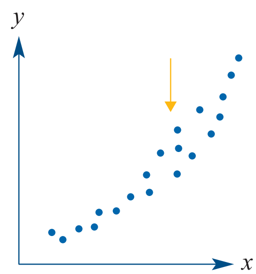




What sorts of non-linear relationships can we linearise using the log y transformation?

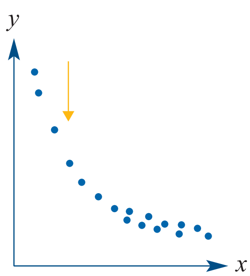
The log y transformation has the effect of compressing the upper end of the y scale. As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the y to log y transformation.

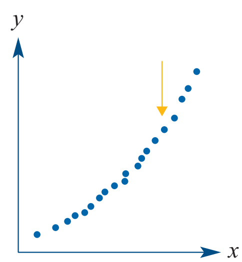




What sorts of non-linear relationships can we linearise using the $1/y$ transformation?

As a guide, relationships that have scatterplots which look like those shown below can often (but not always) be linearised using the y to $1/y$ transformation.





Exercise 6F

The y^2 transformation

1 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	0	2	4	6	8	10
y	1.2	2.8	3.7	4.5	5.1	5.7

b Apply a squared transformation to the y values (y^2). Plot the data, and comment on the form of the relationship between y^2 and x.

c Fit a line to the transformed data with y^2 as the DV and x as the IV. Write its equation. Find the value of y when $x = 9$.

2 a Plot the data in the table. Comment on the relationship between y and x.

x	5	10	15	20	25	30
y	13.2	12.2	11.2	10.0	8.7	7.1

b Apply a squared transformation to the y values (y^2). Plot the data, and comment on the form of the relationship between y^2 and x.

c Fit a line to the transformed data with y^2 as the DV and x as the IV. Write its equation. Find the value of y when $x = 7$.

The log y transformation

3 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	0.1	0.2	0.3	0.4	0.5
y	15.8	25.1	39.8	63.1	100.0

b Apply a log transformation to the y values (log y). Plot the data and comment on the form of the relationship between log y and x.

c Fit a line to the transformed data with log y as the DV and x as the IV. Write its equation. Find the value of log y when $x = 0.6$.

4 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	2	4	6	8	10
y	7.94	6.31	5.01	3.98	3.16

b Apply a log transformation to the y values (log y). Plot the data and comment on the form of the relationship between log y and x.

c Fit a line to the transformed data with log y as the DV and x as the IV. Write its equation. Find the value of log y when $x = 5$.

5 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	1	3	5	7	9
y	7	32	147	681	3162

b Apply a log transformation to the y values (log y). Plot the data, and comment on the form of the relationship between log y and x.

c Fit a line to the transformed data with log y as the DV and x as the IV. Write its equation. Find the value of log y when $x = 2$.

The 1/*x* transformation

6 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	1	2	3	4	5
y	1	0.5	0.33	0.25	0.20

b Apply a reciprocal transformation to the y values ($1/y$). Plot the data and comment on the relationship between $1/y$ and x.

c Fit a line to the transformed data with $1/y$ as the DV and x as the IV. Write its equation. Find the value of y when $x = 2.5$.

7 a Plot the data in the table. Comment on the form of the relationship between y and x.

x	11	14	26	35	41
y	0.43	0.34	0.19	0.14	0.12

b Apply a reciprocal transformation to the y values ($1/y$). Plot the data and comment on the form of the relationship between $1/y$ and x.

c Fit a line to the transformed data with $1/y$ as the DV and x as the IV. Write its equation. Find the value of y when $x = 20$.

d Name a y-axis transformation that should also work for the data. Try it and see.

e Name a y-axis transformation that should not work for the data. Try it and see.

Practical examples

8 The time (in minutes) taken for a local anaesthetic to take effect is related to the dose given. To investigate this relationship a researcher collected the data shown.

Dose	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
Time	3.67	3.55	3.42	3.29	3.15	3.00	2.85	2.68	2.51	2.32	2.12

a Construct a scatterplot showing the relationship between the *dose* of anaesthetic and *time* taken for it to take effect, and comment on the form.

b Apply a squared transformation to the time values (y), again plot the data, and comment on the form of the relationship between time squared (y^2) and dose (x).

c Fit a line to the transformed data with *time*^{2} as the DV and *dose* as the IV. Write its equation. Predict the time for the anaesthetic to take effect when the dose is 0.4 units.

9 The table below shows the number of internet users signing up with a new internet service provider for each of the first nine months of their first year of operation.

Number	24	32	35	44	60	61	78	92	118
Month	1	2	3	4	5	6	7	8	9

a Construct a scatterplot showing the relationship between *number* of users signing up and *month*, and comment on the form. *Month* is the independent variable.

b Apply a log transformation to the number of users (y), again plot the data, and comment on the form of the relationship between log (*number*) and *month* (x).

c Fit a line to the transformed data with log *number* as the DV and *month* as the IV. Write its equation. Predict the value of log *number* for month 10.

10 A group of ten students was given an opportunity to practise a complex matching task as often as they liked before they were assessed on the task. The number of times they practised the task and the number of errors made when assessed are given in the table.

Number of practices	1	2	2	4	5	6	7	7	9	11
Number of errors	14	9	11	5	4	4	3	3	2	2

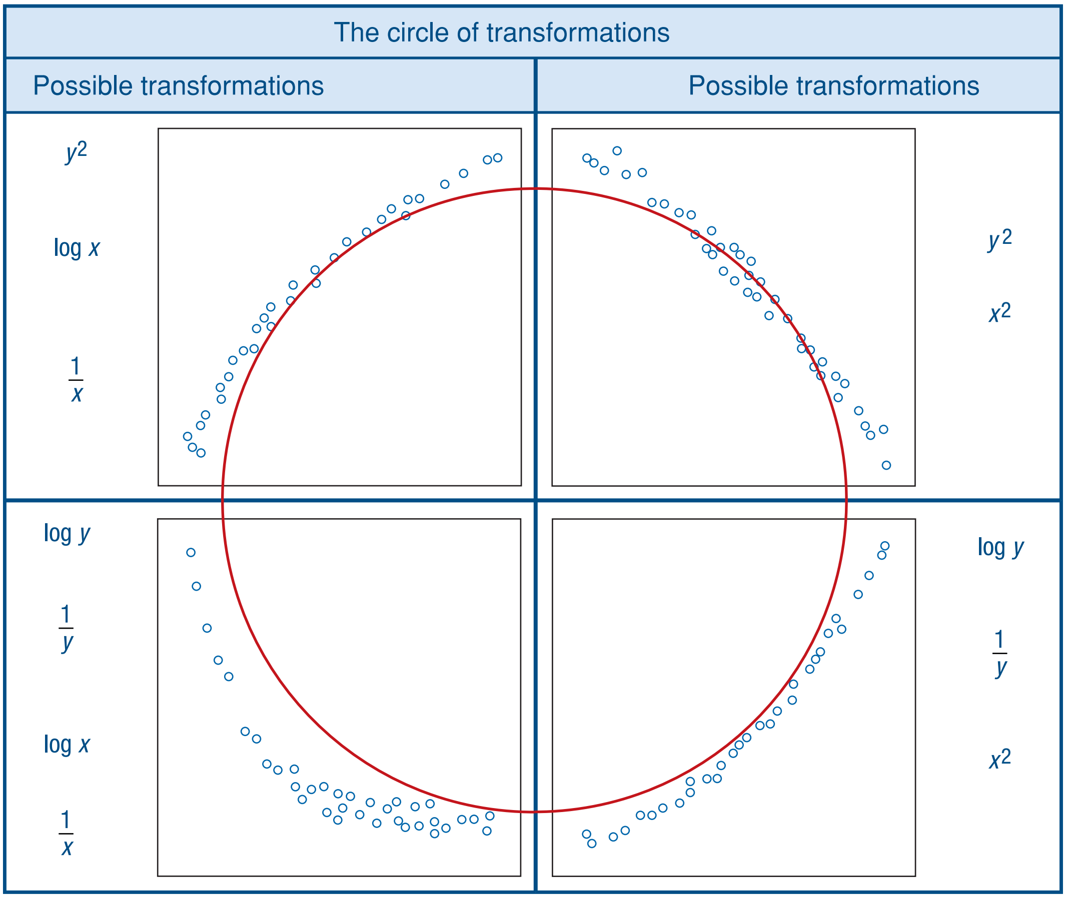
a Construct a scatterplot showing the relationship between *number of practices* and *number of errors* (y), and comment on the form.

b Apply a reciprocal ($1/y$) transformation to the *number of errors* values. Plot the data, and comment on the form of the relationship between $\frac{1}{\textit{number of errors}}$ ($1/y$) and *number of practices* (x).

c Fit a line to the transformed data with $\frac{1}{\textit{number of errors}}$ as the DV and *number of practices* as the IV. Write its equation. Predict the number of errors when the task is practised 12 times.

6.4 Choosing and applying the appropriate transformation

Putting together the information in Sections 6.2 and 6.3, we can see that there may be more than one transformation which linearises the scatterplot. The forms of the scatterplots that can be transformed by the squared, log or reciprocal transformations can be largely classified into one of four categories, shown as the circle of transformations.



Note that the transformations we have introduced in this chapter are able to linearise only those relationships that are consistently increasing or decreasing.

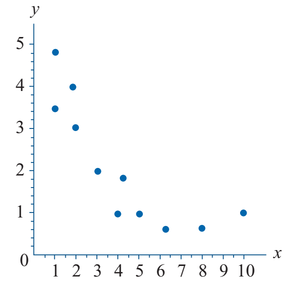
The advantage of having alternatives is that, in practice, we can always try each of them to see which gives us the best result. How do we decide which transformation is the best? The best transformation is the one that results in the best linear model. To choose the best linear model we will consider, for each transformation applied:

- the residual plot, in order to evaluate the linearity of the transformed relationship
- the value of the coefficient of determination (r^2): a higher value indicates a better fit.

This procedure is illustrated in Example 5.

5 The relationship between two variables y and x, as shown in the scatterplot, is non-linear.

Which of the following sets of transformations could possibly linearise this relationship?



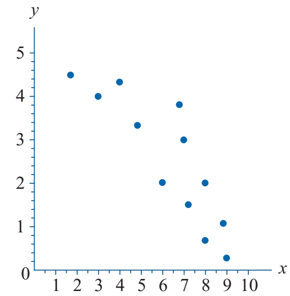
A $\log y, \frac{1}{y}, \log x, \frac{1}{x}$ **B** y^2, x^2

C $y^2, \log x, \frac{1}{x}$ **D** $\log y, \frac{1}{y}, x^2$

E $ax + b$

6 The relationship between two variables y and x, as shown in the scatterplot, is non-linear.

Which of the following transformations is most likely to linearise the relationship?



A a $\frac{1}{x}$ transformation **B** a y^2 transformation

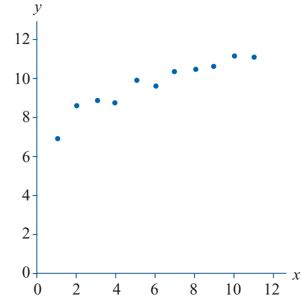
C a $\log y$ transformation **D** a $\frac{1}{y}$ transformation

E a $\log x$ transformation

7 The following data was collected for two related variables x and y.

x	1	2	3	4	5	6	7	8	9	10	11
y	7	8.6	8.9	8.8	9.9	9.7	10.4	10.5	10.7	11.2	11.1

The scatterplot indicates a non-linear relationship. The data is linearised using a $\log x$ transformation. A least squares line is then fitted to the transformed data. The equation of this line is closest to:



A $y = 7.52 + 0.37 \log x$

B $y = 0.37 + 7.52 \log x$

C $y = -1.71 + 0.25 \log x$

D $y = 3.86 + 7.04 \log x$

E $y = 7.04 + 3.86 \log x$

8 Brian has determined from a scatterplot of his data that the appropriate transformations for his data are $\log x$, $1/x$ and y^2. After applying each of these transformations to the data, he obtains the results shown below.

Model	*Residuals*	r^2
y vs x	Curved	79.6%
y vs $\log x$	Random	80.8%
y vs $1/x$	Random	81.9%
y^2 vs x	Random	88.4%

The data below show the age (in years) and diameter at chest height (in cm) of a sample of trees of the same species taken from a commercial plantation.

Age (years)	*Diameter (centimetres)*	*Age (years)*	*Diameter (centimetres)*
4	2.0	16	11.4
5	2.0	18	11.7
8	2.5	22	14.7
8	5.1	25	16.5
8	7.5	29	15.2
10	5.1	30	15.2
10	8.9	34	17.8
12	12.4	38	17.8
13	9.0	40	19.1
14	6.4		

a We wish to predict the age of a tree from its diameter at chest height. In this situation, which is the dependent variable and which is the independent variable?

b Construct a scatterplot and comment on the relationship between age and diameter in terms of direction, outliers, form and strength.

c **i** Fit a linear model to the data and record its equation. Interpret the slope in terms of the problem at hand.

ii Calculate the coefficient of determination and interpret.

iii Form a residual plot and use it to comment on the suitability of modelling the relationship between age and diameter with a straight line.

d Use the x^2 transformation to linearise the data. Then:

i construct a scatterplot of age against diameter squared

ii find the equation of the least squares regression line for the transformed data

iii calculate the coefficient of determination and interpret

iv form a residual plot and use it to comment on the suitability of modelling the relationship between age and diameter squared with a straight line

6 The table below shows the performance level recorded by a number of people on completion of a task, along with the time spent (in minutes) in practising the task.

Time spent on practice	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	7.0
Level of performance	1.0	1.5	2.0	3.0	3.0	3.5	4.0	3.5	3.9	3.6

a Construct a scatterplot showing the relationship between *time spent on practice* and *level of performance*, and comment on the form. The IV is *time*.

b Apply a log transformation to the x values ($\log x$), again plot the data and comment on the form of the relationship between y and $\log x$.

c Fit a line to the transformed data and write down its equation. Use the equation to predict the expected level of performance (correct to one decimal place) for a person who spends 2.5 minutes practising the task.

How to construct a time series using the TI-Nspire CAS

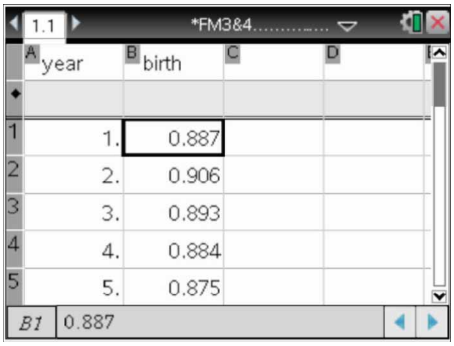
Construct a time series plot for the data presented below. The years have been recoded as 1, 2, . . . , 12, as is common practice.

2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
1	2	3	4	5	6	7	8	9	10	11	12
0.887	0.906	0.893	0.884	0.875	0.861	0.855	0.848	0.846	0.844	0.833	0.848

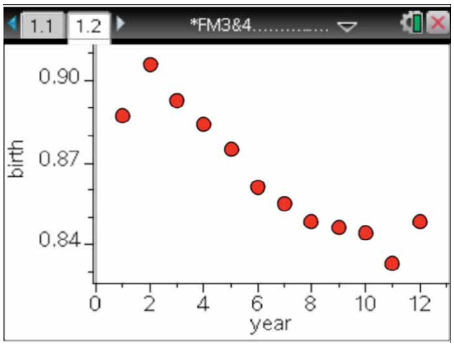
Steps

1 Start a new document by pressing ctrl + **N**.

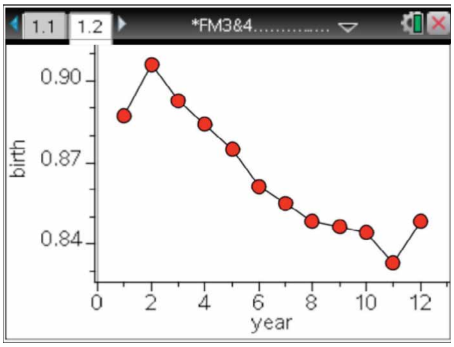
2 Select **Add Lists & Spreadsheet.** Enter the data into lists named *year* and *birth*.



3 Press ctrl + **I** and select **Add Data & Statistics**. Construct a scatterplot of *birth* against *year*.
Let *year* be the independent variable and *birth* the dependent variable.



4 To display as a connected time series plot, move the cursor to the main graph area and press ctrl + menu **>Connect Data Points**. Press enter.



ISBN 978-1-107-65590-4

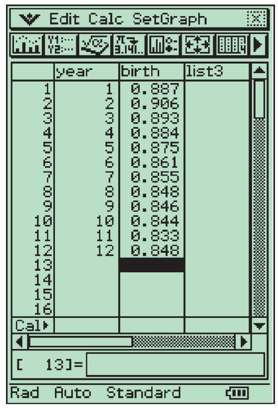
How to construct a time series using the ClassPad

Construct a time series plot for the data presented below. The years have been recoded as 1, 2, . . . 12, as is common practice.

2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
1	2	3	4	5	6	7	8	9	10	11	12
0.887	0.906	0.893	0.884	0.875	0.861	0.855	0.848	0.846	0.844	0.833	0.848

Steps

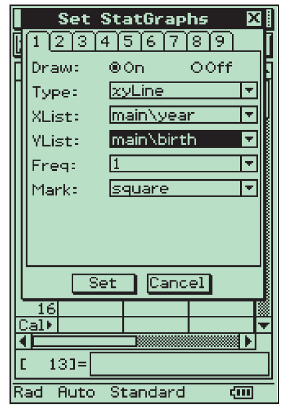
1 Open the **Statistics** application and enter the data into the columns named *year* and *birth*. Your screen should look like the one shown.



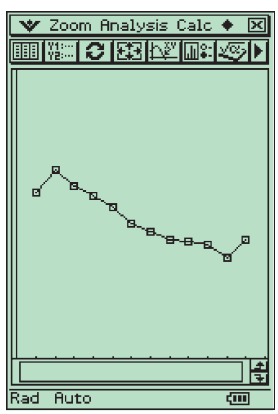
2 Tap [icon] to open the **Set StatGraphs** dialog box and complete as follows.

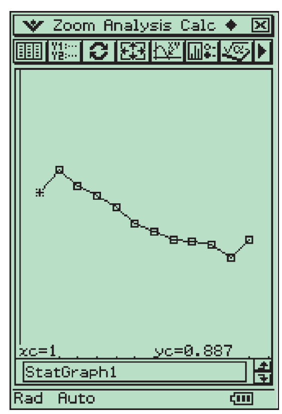
- **Draw:** select **On**
- **Type:** select **xyLine** (▾)
- **XList:** select **main/year** (▾)
- **YList:** select **main/birth** (▾)
- **Freq:** leave as **1**
- **Mark:** leave as **square**

Tap SET to confirm your selections.



3 Tap [icon] in the toolbar at the top of the screen to display the time series plot in the bottom half of the screen. To obtain a full-screen display, tap Resize from the icon panel. Tap [icon] from the toolbar, and use ◀ and ▶ to move from point to point to read values from the plot.



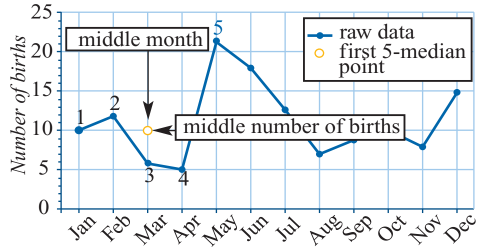


Example 5 **5-median smoothing using a graphical approach**

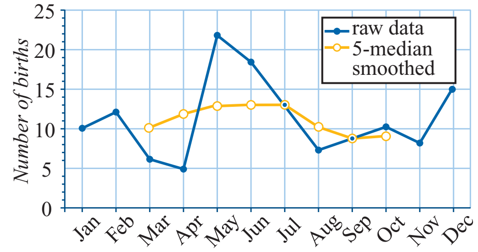
Construct a 5-median smoothed plot of the time series data shown.

Solution

1 Locate on the time series plot the median of the *first* five points (Jan, Feb, Mar, Apr, May), as shown opposite.



2 Then move onto the next five points to be smoothed (Feb, Mar, Apr, May, Jun). The process is then repeated until you run out of groups of five points. The 5-median points are then joined up with line segments to give the final smoothed plot, as shown.



Note: The five-median smoothed plot is much smoother than the three-median smoothed plot.

When smoothing is carried out over an even number of data points, **centring** can be used to align the smoothed values with the original time periods. However, when using the graphical approach to median smoothing, centring is not required.

Exercise 7C

1 Find for the time series data in the table:

t	1	2	3	4	5	6	7	8	9
y	5	2	5	3	1	0	2	3	0

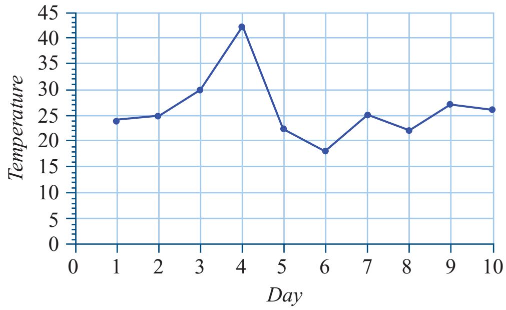
a the 3-median smoothed y value for $t = 4$
b the 3-median smoothed y value for $t = 6$
c the 3-median smoothed y value for $t = 2$
d the 5-median smoothed y value for $t = 3$
e the 5-median smoothed y value for $t = 7$
f the 5-median smoothed y value for $t = 4$

Pages changed in the Enhanced Version

2 Complete the following table.

t	1	2	3	4	5	6	7	8	9
y	10	12	8	4	12	8	10	18	2
3-median smoothed y	–								–
5-median smoothed y	–	–						–	–

3 The time series plot below shows the maximum daily temperatures (in ° Celsius) in a city over a period of 10 consecutive days.



Use the graphical approach to determine the smoothed temperature:

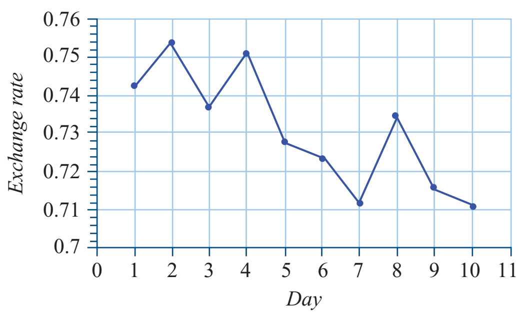
a for day 4 using:

i 3-median smoothing **ii** 5-median smoothing

b for day 8 using:

i 3-median smoothing **ii** 5-median smoothing

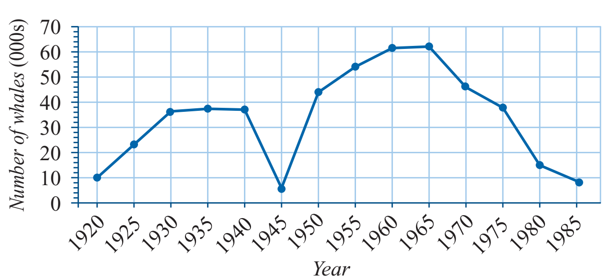
4 The time series plot below shows the value of the Australian dollar in US dollars (Exchange rate) over a period of 10 consecutive days in 2009.



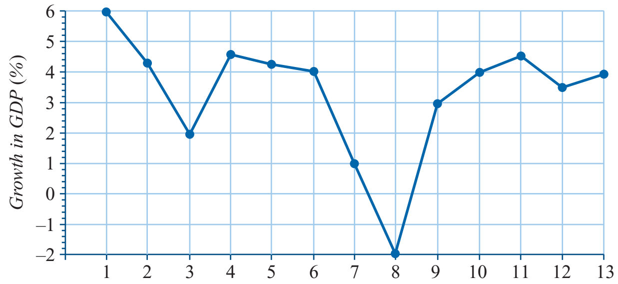
Use 5-median smoothing to graphically smooth the plot and comment on the smoothed plot.

5 Use the graphical approach to smooth the time series plot below using:

a 3-median smoothing **b** 5-median smoothing



6 The time series plot below shows the percentage growth of GDP (gross domestic product) over a 13-year period.



a Smooth the times series graph:
i using 3-median smoothing **ii** using 5-median smoothing

b What conclusions can be drawn about the variation in GDP growth from these smoothed time series plots?

7.4 Seasonal indices

When the data under consideration have a seasonal component, it is often necessary to remove this component by deseasonalising the data before further analysis. To do this we need to calculate **seasonal indices**. Seasonal indices tell us how a particular season (generally a day, month or quarter) compares to the average season.

Consider the (hypothetical) monthly seasonal indices for unemployment given in the table below:

Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*	*Total*
1.1	1.2	1.1	1.0	0.95	0.95	0.9	0.9	0.85	0.85	1.1	1.1	12.0

Seasonal indices are calculated so that their **average** is **1**. This means that the **sum of the seasonal indices** equals the **number of seasons**. Thus, if the seasons are months, the seasonal indices add to 12. If the seasons are quarters, then the seasonal indices would add to 4, and so on.

Interpreting seasonal indices

- The seasonal index for unemployment for the month of February is 1.2.
  Seasonal indices are easier to interpret if we convert them to percentages. Remember, to convert a number to a percentage, just multiply by 100.
  A seasonal index of 1.2 for February, written as a percentage, is 120%.
  A seasonal index of 1.2 (or 120%) tells us that February unemployment figures tend to be 20% **higher** than the monthly average. Remember, the average seasonal index is 1 or 100%.
- The seasonal index for August is 0.90 or 90%.
  A seasonal index of 0.9 (or 90%) tells us that the August unemployment figures tend to be only 90% of the monthly average. Alternatively, August unemployment figures are 10% **lower** than the monthly average.

3 The table below shows the quarterly newspaper sales of a corner store. Also shown are the seasonal indices for newspaper sales for the first, second and third quarters. Complete the table.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Sales	1256	1060	1868	1642
Deseasonalised sales				
Seasonal index	0.8	0.7	1.3	

4 The quarterly cream sales (in litres) made by the same corner store, along with seasonal indices for cream sales for three of the four quarters, are shown in the table below. Complete the table.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Sales	68	102	115	84
Deseasonalised sales				
Seasonal index		1.10	1.15	0.90

5 Each of the following data sets records quarterly sales ($000s). Use the data to determine the seasonal indices for the four quarters. Give your results correct to two decimal places. Check that your seasonal indices add to 4.

a

Q1	Q2	Q3	Q4
48	41	60	65

b

Q1	Q2	Q3	Q4
60	56	75	78

6 Each of the following data sets records monthly sales ($000s). Use the data to determine the seasonal indices for the 12 months. Give your results correct to two decimal places. Check that your seasonal indices add to 12.

a

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
12	13	14	17	18	15	9	10	8	11	15	20

b

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
22	19	25	23	20	18	20	15	14	11	23	30

7 The number of waiters employed by a restaurant chain in each quarter of one year, along with some seasonal indices that have been calculated from the previous year's data, are given in the following table.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Number of waiters	198	145	86	168
Seasonal index	1.30		0.58	1.10

a What is the seasonal index for the second quarter?

b The seasonal index for Quarter 1 is 1.30. Explain what this means in terms of the average quarterly number of waiters.

c Deseasonalise the data.

8 The following table shows the number of students enrolled in a 3-month computer systems training course along with some seasonal indices that have been calculated from the previous year's enrolment figures. Complete the table by calculating the seasonal index for spring and the deseasonalised student numbers for each course.

	Summer	*Autumn*	*Winter*	*Spring*
Number of students	56	125	126	96
Deseasonalised numbers				
Seasonal index	0.5	1.0	1.3	

9 The following table shows the monthly sales figures and seasonal indices (for January to December) for a product produced by the VMAX company.

a Complete the table by:

i calculating the missing seasonal index

ii evaluating the deseasonalised sales figures

b The seasonal index for July is 0.90. Explain what this means in terms of the average monthly sales.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Sales ($000s)	166	215	203	209	178	165	156	256	243	207	165	106
Sales (deseasonalised)												
Seasonal index	1.0		1.1	1.0	1.0	1.0	0.9	1.2	1.2	1.1	1.0	0.7

7.5 Fitting a trend line and forecasting

Fitting a trend line

If there appears to be a linear trend in the data, we can use regression techniques to fit a line to the data. Usually we use the least squares technique but, if there are outliers in the data, the 3-median line is more appropriate.

However, before we use either, we should always check our time series plot to see that the trend is linear. If it is not linear, data transformation techniques should be used to linearise the data first. The next example demonstrates using the least squares regression to fit a trend line to data that have no seasonal component.

Example 9 **Fitting a trend line (no seasonality)**

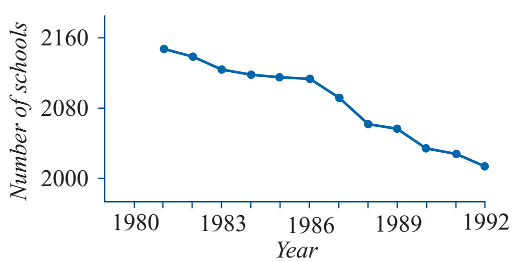
Fit a trend line to the data in the following table, which shows the number of government schools in Victoria over the period 1981–92, and interpret the slope.

Year	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
Number	2149	2140	2124	2118	2118	2114	2091	2064	2059	2038	2029	2013

Solution

1 Construct a time series plot of the data to confirm that the trend is linear:

Note: For convenience we let 1981 = 1, 1982 = 2 and so on when entering the data into a calculator.



2 Use a calculator (with *Year* as the independent variable and *Number of schools* as the dependent variable) to find the equation of least squares regression line.

Number of schools = 2169 − 12.5 × year
Over the period 1981–92 the number of schools in Victoria was decreasing at an average rate of 12.5 per year.

Forecasting

Using a trend line fitted to a time series plot to make predictions about future values is known as **forecasting**.

Example 10 Forecasting (no seasonality)

How many government schools do we predict for Victoria in 2015 if the same decreasing trend continues? Give your answer correct to the nearest whole number.

Solution

Substitute the appropriate value for *year* in the equation determined using least squares regression. Since 1981 was designated as year '1', then 2015 is year '35'.

Number of schools = 2169 − 12.5 × year
= 2169 − 12.5 × 35
= 1732 (correct to the nearest whole number)

Note: As with any relationship, extrapolation should be done with caution!

Taking seasonality into account

When data exhibit seasonality it is a good idea to deseasonalise the data first before fitting the trend line, as shown in the following example.

Example 11 Fitting a trend line (seasonality)

The deseasonalised quarterly sales data from Mikki's shop are shown below.

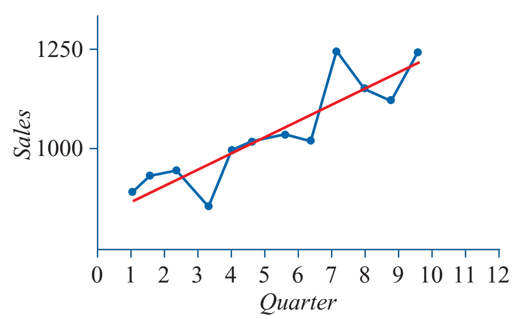
Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Sales	893	943	955	858	1005	1026	1043	1040	1261	1151	1115	1267

Fit a trend line and interpret the slope.

Solution

1 Plot the time series.

2 Using the calculator (with *Quarter* as the IV and *Sales* as the DV), find the equation of the least squares regression line. Plot it on the time series.



d Draw in the trend line on your time series plot.

e Use the trend line to forecast the percentage of retails sales which will be made by departmental stores in Year 15.

5 The average ages of mothers having their first child in Australia over the years 1989–2002 are shown below.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Age	27.3	27.6	27.8	28.0	28.3	28.5	28.6	28.8	29.0	29.1	29.3	29.5	29.8	30.1

a Fit a least squares regression trend line to the data, using 1989 as Year 1, and interpret the slope.

b Use this trend relationship to forecast the average ages of mothers having their first child in Australia in 2018 (Year 30). Explain why this prediction is not likely to be reliable.

6 The sale of boogie boards for a certain company over a 2-year period is given in the following table.

	Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
Year 1	138	60	73	230
Year 2	283	115	163	417

The quarterly seasonal indices are given opposite.

Seasonal index	1.1297	0.4747	0.6248	1.7709

a Use the seasonal indices to calculate the deseasonalised sales figures for this period.

b Plot the actual sales figures and the deseasonalised sales figures for this period and comment on the plot.

c Fit a trend line to the deseasonalised sales data.

d Use the relationship calculated in **c**, together with the seasonal indices, to forecast the sales for the first quarter of Year 4.

7 The sales of motor vehicles for a large car dealer over a 4-year period and the quarterly seasonal indices are given in the tables opposite.

	Quarter 1	*Quarter 2*	*Quarter 3*	*Quarter 4*
Year 1	202	396	274	238
Year 2	212	350	246	238
Year 3	241	453	362	355
Year 4	253	471	389	325

Seasonal index	0.7314	1.3400	1.0091	0.9196

a Use the seasonal indices to calculate the deseasonalised sales figures for this period.

b Plot the actual sales figures and the deseasonalised sales figures for this period and comment on the plots.

c Fit a trend line to the deseasonalised sales data.

d Use the relationship calculated in **c**, together with the seasonal indices, to forecast the sales for the fourth quarter of Year 5.

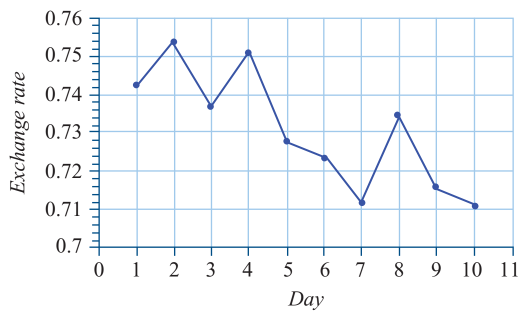
8 The median duration of marriage to divorce (years) in Australia over the years 1992–2002 is given in the following table.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Duration	10.5	10.7	10.9	11.0	11.0	11.1	11.2	11.3	11.6	11.8	12.0

a Fit a least squares regression trend line to the data, using 1992 as Year 1, and interpret the slope.

b Use this trend relationship to forecast the median duration of marriage to divorce in Australia in 2015. Explain why this prediction is not likely to be reliable.

9 The time series plot below shows the value of the Australian dollar in US dollars (Exchange rate) over a period of 10 consecutive days in 2009.



Fit a three-median line to the time series plot. Determine and interpret its slope in terms the variables *exchange rate* and *day*.

Extended-response questions

1 The infant mortality rate (number of deaths of infants aged under one year per 100 000 live births) in Victoria over the period 1990–2002 is given in the following table.

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Mortality rate	523	428	366	347	327	308	308	300	283	331	268	284	305
3-moving mean													
3-moving median													

a Use 3-moving mean and 3-moving median smoothing to complete the table. (Give your answers to the nearest whole number.)

b Plot the original data, together with the mean and median smoothed data, and comment on the plots.

2 The table below shows the average mortgage interest rate for the period 1987–97.

Year	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Interest rate	15.50	13.50	17.00	16.50	13.00	10.50	9.50	8.75	10.50	8.75	7.55
3-moving mean											

a Construct a time series plot for average mortgage interest rate during the period 1987–97.

b Use the 3-moving mean method to complete the table.

c Plot the smoothed interest rate data and comment on any trend revealed.

d Fit a trend line to the data and find its equation (with 1987 as Year 1). Interpret the slope.

e What interest rate was predicted by the trend line for 1998? In making this prediction, was the forecaster interpolating or extrapolating?

f When does the trend predict that the interest rates will fall to zero? Do you think that this will ever happen? Why? What assumption are we making in our prediction that will probably not hold true in the future?

3 The sales of pies are known to be seasonal. The pie manufacturer has produced the following quarterly seasonal indices for the pie sales.

Q1	Q2	Q3	Q4
0.6	1.2	1.4	0.8

The trend equation for deseasonalised data is:

$Sales = 12000 + 100 \times Quarter\ number$

a Estimate the (deseasonalised) quarterly sales for the second quarter of Year 2, if the first quarter of Year 1 is Quarter number 1.

b Use the appropriate seasonal index to obtain a forecast for the second quarter of Year 2.

c The season index for Quarter 1 is 0.60. Explain what this means in terms of quarterly sales.

4 The sales of ice-cream are known to be seasonal. The ice-cream manufacturer has produced the following quarterly seasonal indices for the sales of ice-cream.

Q1	Q2	Q3	Q4
1.5	0.7	0.6	1.2

The trend equation for deseasonalised data is:

$Sales = 10\,000 + 80 \times Quarter\ number$

a Estimate the (deseasonalised) quarterly sales for the third quarter of Year 2, if the first quarter of Year 1 is Quarter number 1.

b Use the appropriate seasonal index to obtain a forecast for the third quarter of Year 2.

c If ice-cream sales in the fourth quarter of 2011 were reported as 18 564, determine the deseasonalised sales for this quarter.

5 The seasonal indices for the four quarters for a particular product have been calculated from sales data over many years. The data give quarterly sales for Year 1.

Season	*Summer*	*Autumn*	*Winter*	*Spring*
Year 1	1976	2940	3195	4900
Seasonal index	0.80	1.05	0.90	1.25

a Calculate the deseasonalised sales figure for summer.

b A least squares regression trend line has been fitted to the deseasonalised sales figures. The equation of the trend line is:

$Sales = 1910 + 510 \times Time\ period$

where summer, Year 1, is time period 1.

i Estimate the (deseasonalised) quarterly sales for the spring of Year 3.

ii Use the seasonal index to obtain a better forecast for the spring of Year 3.

c The seasonal index for spring is 1.25. Explain what this means in terms of the quarterly sales.

19 The relationship between y and x revealed in the scatterplot is clearly non-linear. In an attempt to transform the data to linearity, you could use:

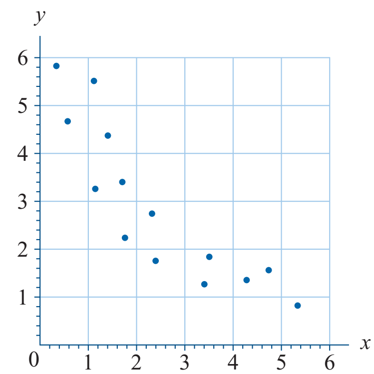
A a log x transformation **B** a log y transformation **C** an x^2 transformation

D a $\frac{1}{x}$ transformation **E** a $\frac{1}{y}$ transformation

20 The following data was collected for two related variables x and y.

x	0.4	0.5	1.1	1.1	1.2	1.6	1.7	2.3	2.4	3.4	3.5	4.3	4.7	5.3
y	5.8	4.7	3.3	5.5	4.2	3.4	2.3	2.8	1.8	1.3	1.9	1.2	1.6	0.9

The scatterplot indicates a non-linear relationship. The data is linearised using a reciprocal transformation ($\frac{1}{y}$). A least squares line is then fitted to the transformed data. The equation of this line is closest to:



A $\dfrac{1}{y} = 0.08 + 0.16x$

B $\dfrac{1}{y} = 0.16 + 0.08x$

C $\dfrac{1}{y} = -0.08 + 5.23x$

D $\dfrac{1}{y} = 5.23 - 0.08x$

E $\dfrac{1}{y} = 1.44 + 1.96x$

8.4 Time series

1 The quarterly sales figures for a soft drink company and the seasonal indices are as shown.

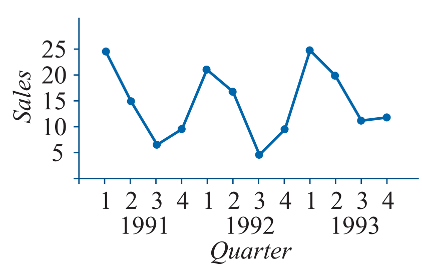
Quarter	1	2	3	4
Sales ($000s)	2600	2200	2100	2800
Seasonal index	1.05	0.95	0.8	1.2

The deseasonalised figure (in $000s) for Quarter 3 is:

A 1680 **B** 2100 **C** 2425 **D** 2625 **E** 2800

[VCAA pre 2006]

2 Which of the following statements best describes the time series represented by the graph?



A This time series shows a seasonal pattern but no linear trend.

B This time series shows a linear trend but no seasonal pattern.

C This time series shows a seasonal pattern and a linear trend.

D This time series shows neither a seasonal pattern nor a linear trend.

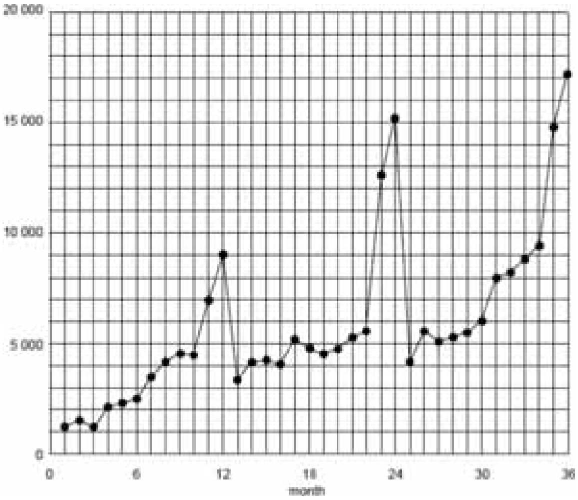
E It is impossible to tell from this information whether this time series exhibits either a seasonal pattern or a linear trend.

[VCAA pre 2006]

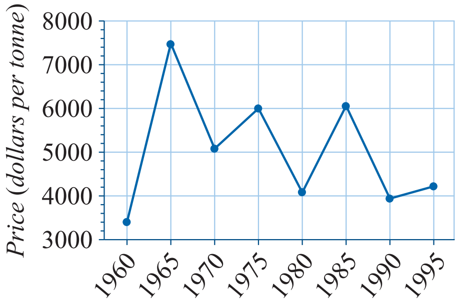
13 The time series plot shows the revenue from sales (in dollars) each month made by a Queensland souvenir shop over a three-year period. A three-median trend line is fitted to this data. Its slope (in dollars per month) is closest to:

A 125 **B** 146 **C** 167 **D** 188 **E** 255

[VCAA 2007]

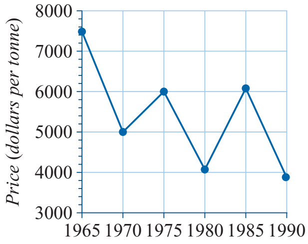


14 The time series plot opposite shows the price (dollars per tonne) of copper ore over the period 1960 to 1995.

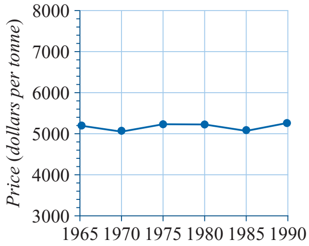


When smoothed, using 3-point median smoothing, the time series plot will look most like:

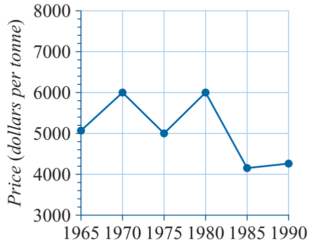
A



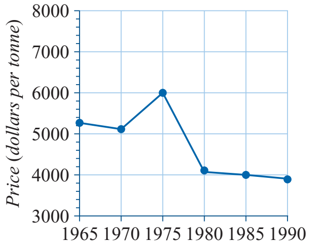
B



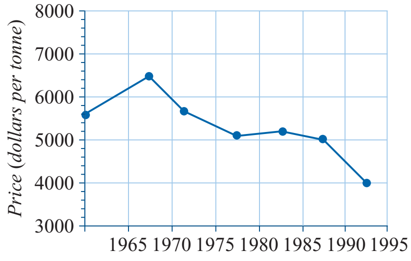
C



D



E



[VCAA pre 2006]

15 The following table gives the number of births in a country hospital over an 8-year period.

Year	1	2	3	4	5	6	7	8
Number of births	99	74	88	103	92	110	109	118

Using the 2-point moving mean method, with centring, the smoothed value of the number of babies born in Year 6 is:

A 101 **B** 103.33 **C** 105.25 **D** 109.5 **E** 110

8.5 Extended-response questions

1 The data in Table 8.1 is based on a study of dolphin behaviour. In this study, the main activities of dolphins observed in the wild were classified as 'travelling', 'feeding' and 'socialising'. The time of observation was recorded as 'morning', 'noon', 'afternoon' or 'evening'.

Table 8.1 Number of dolphins observed by activity and time of observation

	Time of observation			
Activity	*morning*	*noon*	*afternoon*	*evening*
travelling	6	6	14	13
feeding	28	4	0	56
socialising	38	5	9	10

a Complete the following sentences.

i The number of dolphins observed feeding at noon is ______.

ii The dolphin activity most frequently observed in the morning is ______.

To test the assertion that dolphin activity is associated with time of day, the table was percentaged by calculating the appropriate column percentages as displayed in Table 8.2.

Table 8.2 Percentage of dolphins observed by activity and time of observation

	Time of observation			
Activity	*morning*	*noon*	*afternoon*	*evening*
travelling	8	40	61	16
feeding	39	27	0	71
socialising	53	33	39	13

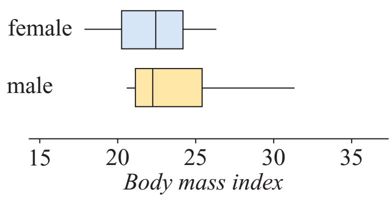
b Use the information in Table 8.2 to describe briefly any relationship that you can see between dolphin activity and the recorded time of observation. Quote appropriate percentages to support your description.

In another study of animal behaviour, investigators collected information on the average hours that various animal species spend in dreaming and non-dreaming sleep. The data for a selected group of 14 of these animals is shown in Table 8.3.

Revision

g Do the data support the contention that, for this sample, weight rating is associated with gender? Justify your answer by quoting appropriate percentages.

h The parallel box plots have been constructed to compare the distribution of BMI for males and females in this sample.



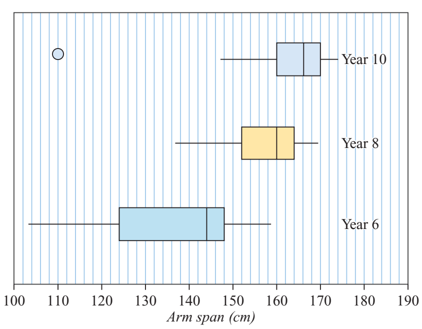
i Use the parallel box plots to identify and name two similar properties of the BMI distributions for males and females.

ii Use the information in the table to determine the mean BMI for the males in this sample. Write your answer correct to one decimal place.

iii The median BMI for males is 22.5. Of the mean or median, which measure gives a better indication of the typical BMI for males? Explain your answer.

[VCAA pre 2006]

4 The arm spans (in cm) were recorded for each of the girls in Years 6, 8 and 10. The results are summarised in the three parallel box plots displayed below.



a Complete the following sentence. The middle 50% of Year 6 students have an arm span between ☐ and ☐ cm.

b The three parallel box plots suggest that arm span and year level are associated. Explain why.

c The arm span of 110 cm of a Year 10 girl is shown as an outlier on the box plot. This value is an error. Her real arm span is 140 cm. If the error is corrected, would this girl's arm span still show as an outlier on the box plot? Give reasons for your answer, showing an appropriate calculation. [VCAA 2008] the common difference is +5, whereas in the sequence

$$19, 17, 15, 13, 11, \ldots$$

the common difference is −2.

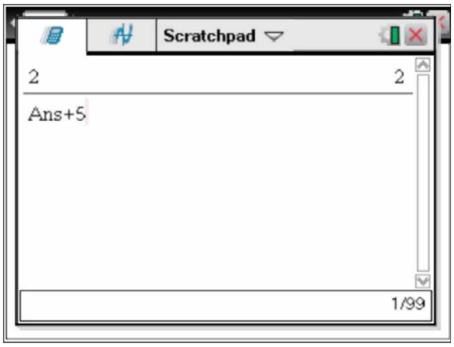
Once you know the first term in an arithmetic sequence and its common difference, the rest of the terms in the sequence can be readily generated. If you want to generate a large number of terms, your graphics calculator will do this with little effort.

How to generate the terms of an arithmetic sequence using the TI-Nspire CAS

Generate the first five terms of the arithmetic sequence: 2, 7, 12, 17, 22, . . .

Steps

1 Press (on) (or (off/on) then (home) on the Clickpad). Then press [A] to open the **Scratchpad:Calculate**. Pressing (scratchpad) also opens the **Scratchpad**.
See the Appendix for more details on the **Scratchpad**.
Note: You can also use (on)**>Documents> New Document>Add Calculator** if preferred.

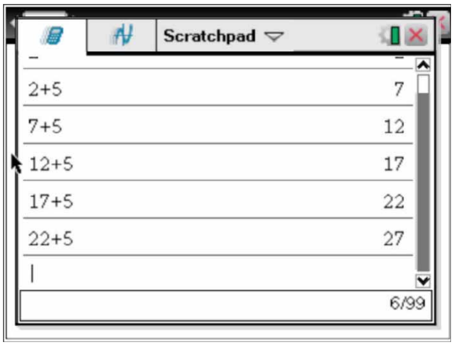


2 Enter the value of the first term, **2**.

3 The common difference for the sequence is 5. So, type **+5**. Press (enter). The second term in the sequence, **7**, is generated.

4 Pressing (enter) again generates the next term, **12**.

5 Keep pressing (enter) until the desired number of terms is generated.



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(Examples 3 and 4). However, the next type of example is most efficiently solved using a graphics calculator.

As often happens, there are several graphics calculator methods that can be used. The method we have chosen uses sequence mode. This is perhaps not the quickest and easiest method. However, it has the advantage of being the only method that works for *all* the problems you will meet in this module. It is therefore worth your while learning it now.

How to generate the terms of a sequence using the TI-Nspire CAS

Generate the terms in an arithmetic sequence with $a = 10$ and $d = 4$.

Steps

Strategy: Find an expression for the nth term of the sequence, as for Example 5. A graphics calculator can then be used to display the sequence in a table.

1 For this sequence, $a = 10$ and $d = 4$.

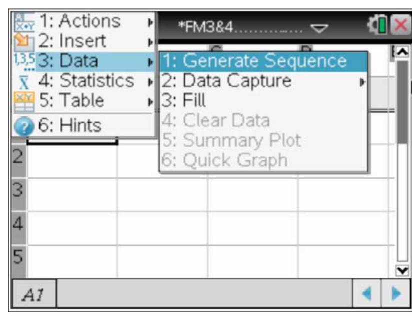
$a = 10,\ d = 4$

2 Use $t_n = a + (n - 1)d$ to write down an expression for the nth term, t_n. Don't simplify.

$t_n = 10 + (n - 1) \times 4$

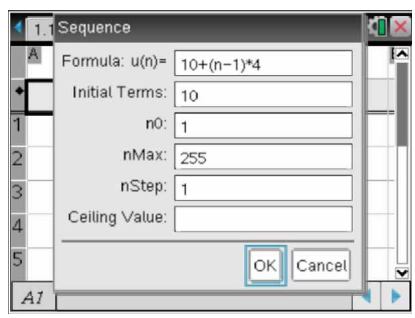
3 Start a new document by pressing (ctrl) + **N**. Select **Add Lists & Spreadsheet**.

a Place the cursor in any cell in column A and press (menu) **>Data** to generate the screen opposite.

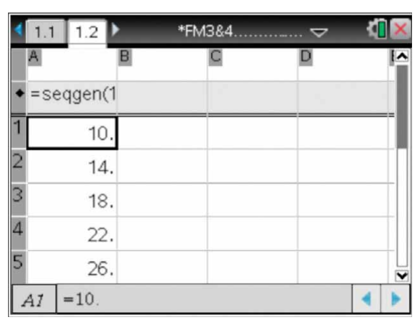


b With the cursor on **Generate Sequence**, press (enter) to display the pop-up screen shown opposite. Type in the entries as shown. Use (tab) to move between entry boxes. Leave the **Max No. Terms** at 255.

Note: The calculator uses u(n) for the nth term.



c Press (enter) to close the pop-up screen and display the sequence of terms. The term number can be read directly from the row number (left-hand side) of the spreadsheet. For example, the 5th term would be 26. Use the ▼ arrow to move down through the sequence to see further terms.



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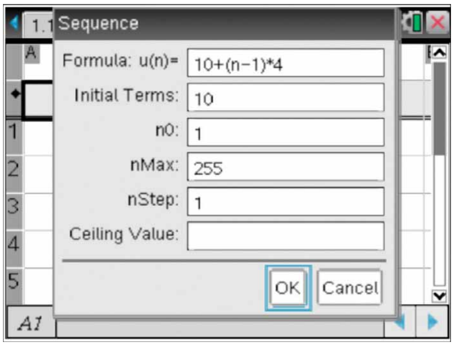
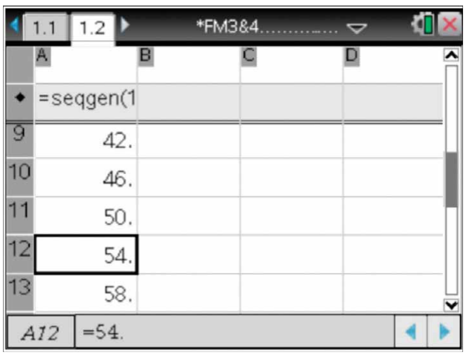
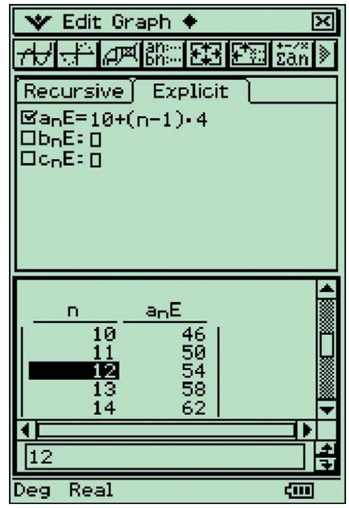
2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term that exceeds 51; in this case, the 12th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	10	11	12	...
t_n	10	14	...	46	50	54	...

The first term to exceed 51 is t_{12}.

Application

Any situation where you start with a fixed amount and add or subtract a fixed amount at regular intervals can be modelled by an arithmetic sequence. For example, the increase in weight of a bag of apples as additional apples are added to the bag or the amount of wine left in a bottle as glasses of wine are poured. The following example involves a person on a weight-loss program.

Example 8 **Application of the *n*th term of an arithmetic sequence**

Before starting on a weight-loss program a man weighs 124 kg. He plans to lose weight at a rate of 1.5 kg a week until he reaches his recommended weight of 94 kg.

a Write down a rule for the man's weight, W_n, at the start of week n.

b If he keeps to his plan, how many weeks will it take the man to reach his target weight of 94 kg?

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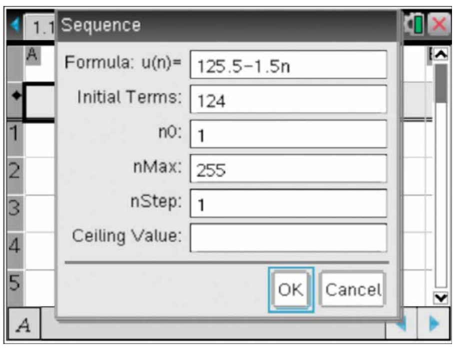
Solution

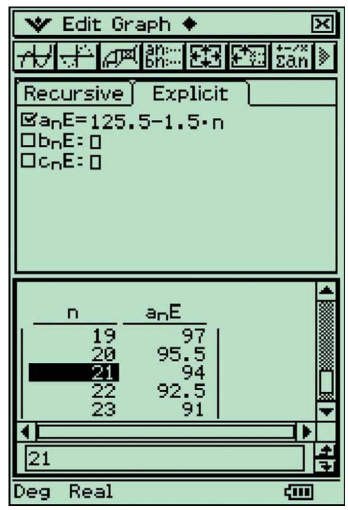
Strategy: You need to recognise that by losing a constant amount of weight each week, the man's weekly weight follows an arithmetic sequence. Using this information, you can write down an expression for his weight in the *n*th week. You can then use this expression to display the sequence of weights in a table and hence determine when the target weight is reached.

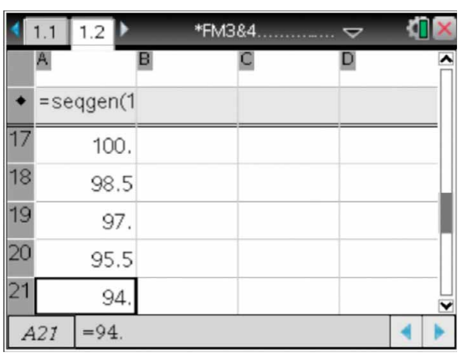
1 Arithmetic sequence with $a = 124$ and $d = -1.5$
Use the rule $W_n = a + (n - 1)d$ to write down an expression for W_n.

Arithmetic sequence
$a = 124,\ d = -1.5$
$W_n = 124 + (n - 1) \times (-1.5)$
$= 124 - 1.5n + 1.5$
$\therefore W_n = 125.5 - 1.5n$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term that is 94 or less; in this case, the 21st term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	…	19	20	21	…
W_n	124	122.5	…	97	95.5	94	…

If the man keeps to his plan, he will reach his target weight by the start of week 21, or after 20 weeks of being on the program.

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Example 11 **Finding when the sum of a sequence first exceeds a given value**

How many terms are required for the sum of the arithmetic sequence 5, 15, 25, . . . to first exceed 200?

Solution

Strategy: Find an expression for the sum of n terms of the sequence. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 200 can then be found.

1 For this sequence, $a = 5$ and $d = 10$. Use this information and the rule $S_n = \frac{n}{2}[2a + (n - 1)d]$ to find an expression for the sum.

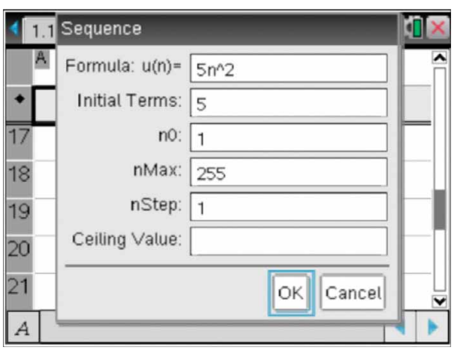
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

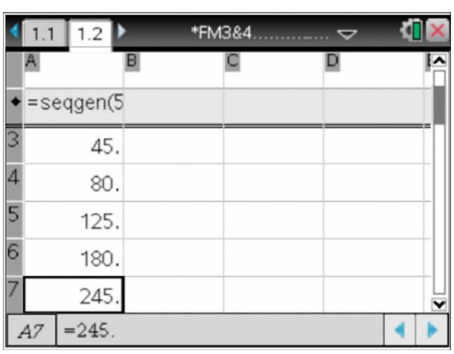
$$a = 5,\ d = 10$$

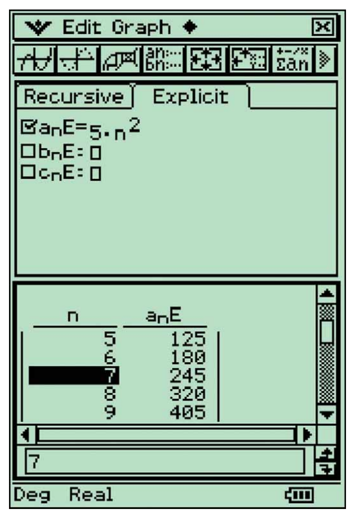
$$\therefore S_n = \frac{n}{2}[2 \times 5 + (n - 1)10]$$

$$= \frac{n}{2}(10 + 10n - 10) = 5n^2$$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the first term in the sequence that exceeds 200; in this case, the 7th term.







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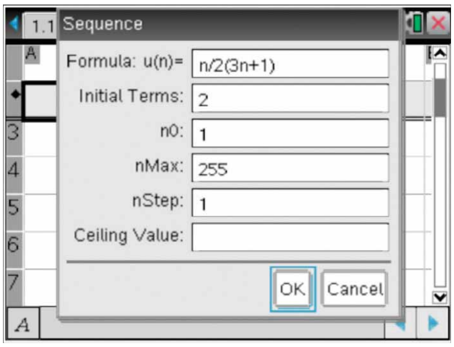
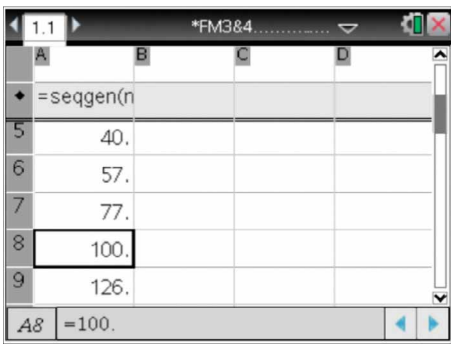
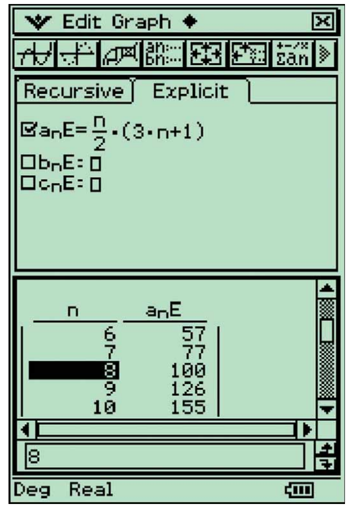
2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that is 100 or less; in this case, the 8th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	7	8	9	...
S_n	2	7	...	77	100	126	...

The girl can build a pattern with eight rows of blocks.

Exercise 9D

1 Use the rule to find the sum of the first:

- **a** six terms of an arithmetic sequence with $a = 5$ and $d = 3$
- **b** five terms of an arithmetic sequence with $a = 12$ and $d = -2$
- **c** seven terms of an arithmetic sequence with $a = -5$ and $d = 5$
- **d** four terms of an arithmetic sequence with $a = 0.1$ and $d = 0.2$
- **e** six terms of an arithmetic sequence with $a = -9$ and $d = 3$

In each case, write out the series and add up the terms to check your answer.

2 Use the rule to find the sum of the arithmetic sequence:

a	4, 8, 12, ...	to 20 terms	**b**	10, 7, 4, ...	to eight terms
c	120, 110, 100, ...	to nine terms	**d**	1, 2, 3, ...	to 1000 terms
e	1.000, 1.005, 1.010, ...	to 100 terms	**f**	−8, −6, −4, ...	to 15 terms

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In other cases, the value of the common ratio is not so obvious. In these cases we can determine the value of the common ratio by dividing one of the terms in the sequence by its immediate predecessor.
For example, the common ratio for the geometric series

$$0.8, 0.32, 0.128, \ldots$$

is

$$r = \frac{0.32}{0.8} = 0.4 \qquad \left(\text{or } \frac{0.128}{0.32} = 0.4\right)$$

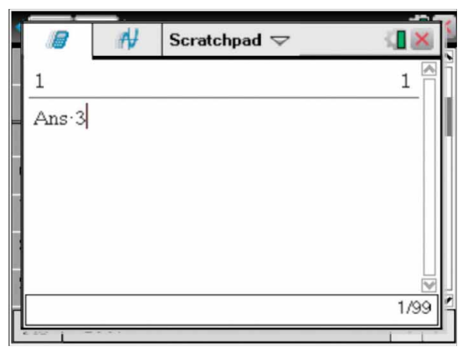
Once you know the first term in a geometric sequence and its common ratio, the rest of the terms in the sequence can be readily generated. If you want to generate a large number of terms, your graphics calculator will do this with little effort.

How to generate the terms of a geometric sequence using the TI-Nspire CAS

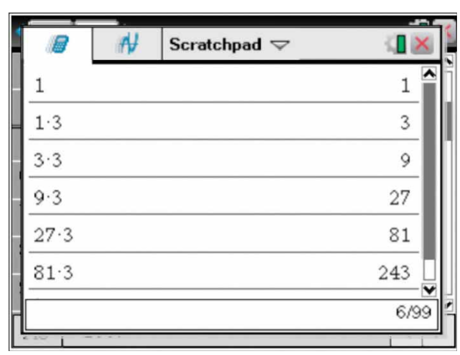
Generate the first five terms of the geometric sequence 1, 3, 9, 27, . . .

Steps

1. Press (⌂ on) (or (off on) then (⌂) on the Clickpad). Then press [A] to open the **Scratchpad:Calculate**. Pressing (calculator key) also opens the **Scratchpad**.



2. Enter the value of the first term, **1**. Press (enter).
3. The common ratio for the sequence is 3. So, type × **3**. Press (enter). The second term in the sequence, 3, is generated.
4. Pressing (enter) again generates the next term, 9.
5. Keep pressing (enter) until the desired number of terms is generated.



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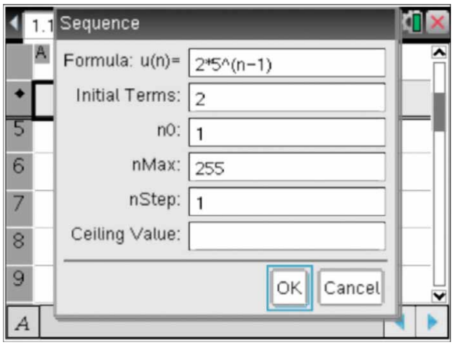
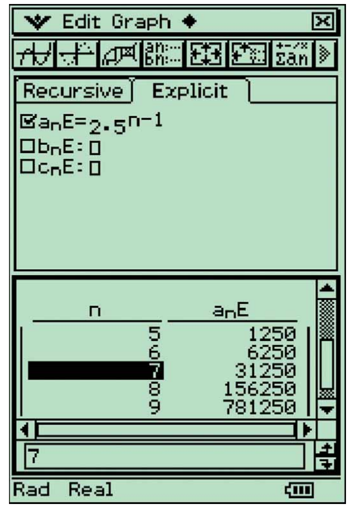
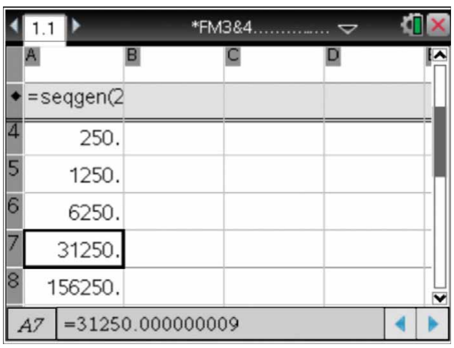
2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that exceeds 30 000; in this case, the 7th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	5	6	7	...
S_n	2	10	...	1250	6250	31250	...

The 7th term is the first to exceed 30 000.

Exercise 9F

1 For the geometric sequence:

a 3, 6, 12, ... write down the values of a, r, and t_3

b 15, 45, 135, ... write down the values of a, r, and t_2

c 200, 100, 50, 25, ... write down the values of a, r, and t_4

d 130, 13, 1.3, ... write down the values of a, r, and t_1

e 20, −24, 28.8, −34.56, ... write down the values of a, r, and t_3

f −120, 30, −7.5, 1.875, ... write down the values of a, r, and t_4

Hint: As you are told these are geometric sequences, you can use any two terms to find r.

2 For a geometric sequence with:

a $a = 20$ and $r = 10$, determine the value of the 3rd term

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Example 24 Using the rule to find the sum of a geometric sequence

For the geometric sequence 2, 8, 32, . . . , find an expression for the sum of the first n terms.

Solution

1 For this sequence, $a = 2$ and $r = 4$.

$a = 2,\ r = 4$

2 As $r > 1$, use the rule $S_n = \dfrac{a(r^n - 1)}{r - 1}$

3 Substitute these values to obtain an expression for the sum of n terms. Simplify.

$$\therefore S_n = \frac{2(4^n - 1)}{4 - 1} = \frac{2(4^n - 1)}{3}$$

4 Write down your answer.

The sum of the first n terms is $\dfrac{2(4^n - 1)}{3}$.

Example 25 Finding when the sum of a sequence first exceeds a given value

When does the sum of the geometric sequence 2, 10, 50, . . . first exceed 7500?

Solution

Strategy: Find an expression for the sum of n terms of the sequence. A graphics calculator can then be used to display the sequence in a table. The first term that exceeds 7500 can then be found.

1 Use the rule $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

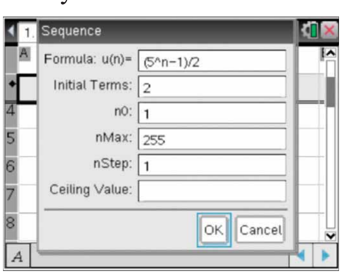
In this example, $a = 2, r = 5$ $(10 \div 2)$.

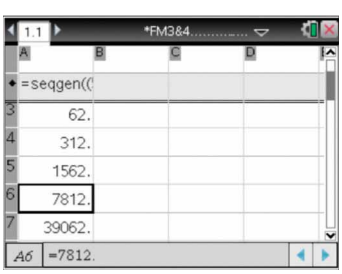
$a = 2, r = 5$

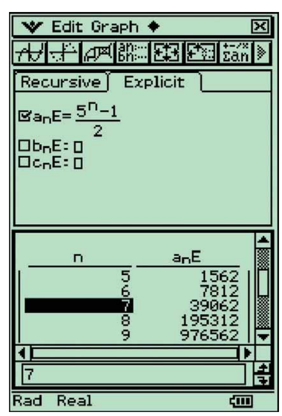
Use this information and the rule to find an expression for the sum in terms of n.

$$\therefore S_n = \frac{2(5^n - 1)}{5 - 1} = \frac{(5^n - 1)}{2}$$

2 Use your calculator to generate the sequence of terms and move down through this sequence of terms until you find the last term in the sequence that exceeds 7500; in this case, the 6th term.







3 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	2	...	4	5	6	...
S_n	2	12	...	312	1562	7812	...

The sum of the sequence first exceeds 7500 after adding six terms.

Exercise 9H

1 Use the rule to find the sum of the first:

a five terms in a geometric sequence with $a = 5$ and $r = 3$
b four terms in a geometric sequence with $a = 10$ and $r = -0.1$
c three terms in a geometric sequence with $a = -5$ and $r = 1.2$
d three terms in a geometric sequence with $a = 10\ 000$ and $r = -1.06$
e five terms in a geometric sequence with $a = 512$ and $r = 0.5$

In each case, write out the terms and add to check your answer.

2 Use the rule to find the sum of the geometric sequence:

a 2, 4, 8, . . . to 20 terms
b 1000, 100, 10, . . . to 15 terms
c 1, 1.2, 1.44, . . . to 9 terms
d 2, 1, 0.5, . . . to 20 terms
e 1.000, 1.05, 1.1025, . . . to 10 terms
f 1.1, 1.21, 1.331, . . . to 5 terms

3 Use the rule to write down an expression for the sum of the first n terms of the following:

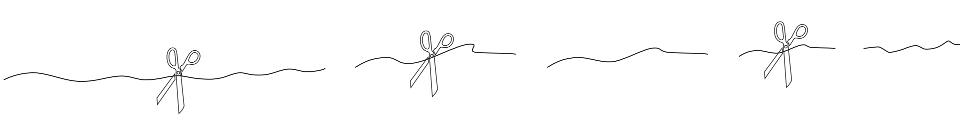
a a geometric sequence with $a = 10$ and $r = 1.5$
b a geometric sequence with $a = 50$ and $r = 0.2$
c 4, 20, 100, . . .
d 8, 4, 2, . . .
e 0.9, 0.3, 0.1, . . .

4 When does the sum of the geometric sequence:

a 2, 4, 8, . . . first exceed 50?
b 10, 100, 1000, . . . first exceed 1 000 000?
c 0.25, 0.5, 1, . . . first exceed 10?
d 2000, 2100, 2205, . . . first exceed 100 000?
e 0.1, 0.5, 2.5, . . . first exceed 12?
f 0.1, 1, 10, . . . first exceed 1000?

9.9 The sum of an infinite geometric sequence

Imagine taking a piece of string of length one metre, and cutting it in half. Put one piece aside and cut the other piece in half again. Keep on repeating the process.



How to graph the terms of a sequence using the TI-Nspire CAS

Plot the terms of the following sequences on the same graph:

- sequence 1: arithmetic with $a = 2$ and $d = 2$
- sequence 2: geometric with $a = 2$ and $r = 2$

for $n = 1, 2, \ldots, 6$. These are the sequences plotted previously.

Steps

1 Write an expression for the nth term of the two sequences using the rules.

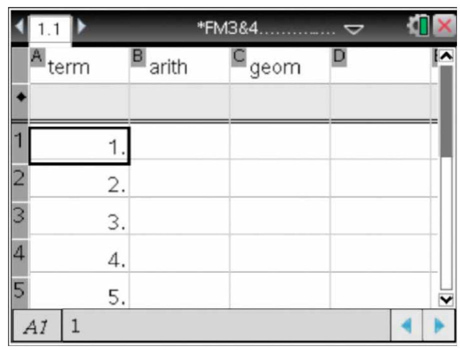
arithmetic: $t_n = 2 + (n - 1) \times 2$
geometric: $t_n = 2 \times 2^{(n-1)}$

2 Start a new document by pressing ctrl + **N**.

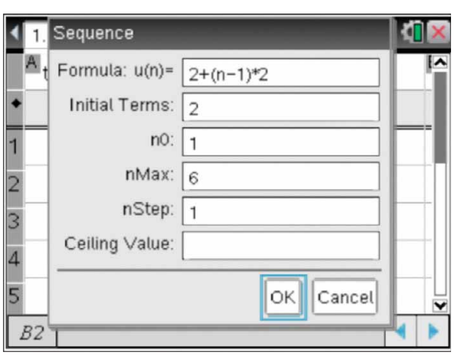
3 Select **Add Lists & Spreadsheet**.

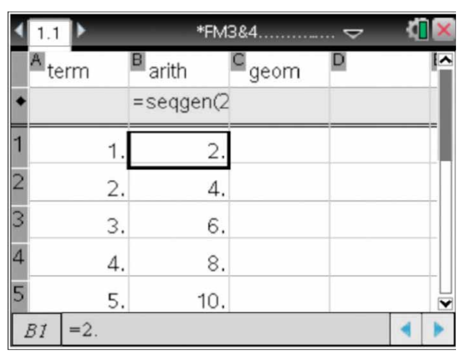
a Enter the numbers **1** to **6** into a list named *term*, as shown.
Note: You can also use the sequence command to do this.

b Name column B *arith* and column C *geom*.



4 **a** Place the cursor in any cell in column B and press menu **>Data>Generate Sequence** and type in the entries as shown (below left). Use tab to move between entry boxes. Press enter to close the pop-up screen and display the values of the first six terms in the arithmetic sequence in column B (below right).





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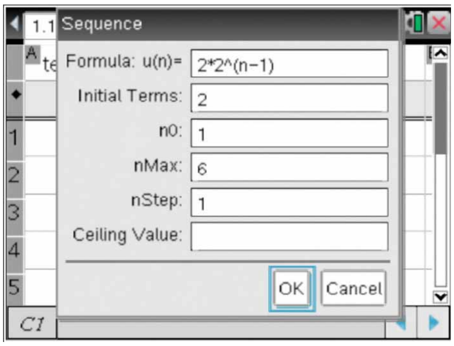
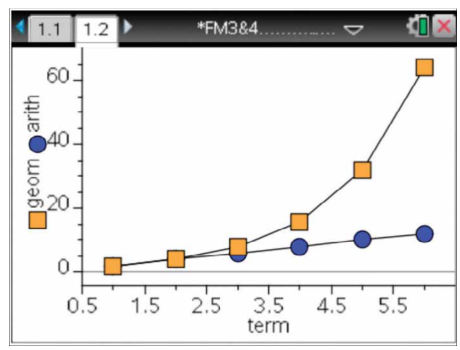
b Place the cursor in any cell in column C and press (menu) **>Data>Generate Sequence** and type in the entries shown (below left). Press (enter) to close the pop-up screen and display the values of the first six terms in the geometric sequence in column C (below right).



5 Press (ctrl) + [I] and select **Add Data & Statistics**.

a Construct a scatterplot using *term* as the independent variable and *arith* as the dependent variable. To connect points, press (menu) **>Plot Properties> Connect Data Points**.

b Press (menu) **>Plot Properties>Add Y Variable**. Select the variable *geom*. Press (enter). This will show the terms of the geometric sequence on the same plot.



Notes:

1 The arithmetic sequence increases in a linear manner, whereas the geometric sequence increases in an exponential manner.

2 Sequence plotting can also be done in the **Graphs** application.

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14 In an arithmetic sequence, $a = 250$ and $d = 26$. The first term in this sequence to exceed 500 is the:

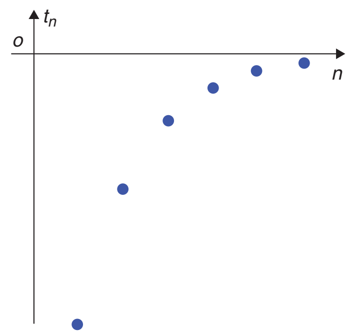
A 8th term **B** 9th term **C** 10th term **D** 11th term **E** 12th term

15 If successive terms in a geometric sequence increase by 12%, then the common ratio, r, is:

A 0.12 **B** 0.88 **C** 1.0 **D** 1.12 **E** 1.2

16 The graph opposite shows the first six terms of a geometric sequence. Its common ratio, r, could be:

A $r = -1.5$ **B** $r = -0.5$
C $r = -0.25$ **D** $r = 0.5$
E $r = 1.5$



17 The first four terms of a geometric sequence are: 10, −30, 90, −270. The sum of the first eight terms of this sequence is:

A −32 000 **B** −16 400 **C** 16 400 **D** 16 405 **E** 32 800

18 Before he began training, Jethro's longest jump was 5.80 m.
After the first month of training, his longest jump had increased by 0.32 m.
After the second month of training, his longest jump had increased by another 0.24 m.
After the third month of training, his longest jump had increased by another 0.18 m.
If this pattern of improvement continues, Jethro's longest jump, correct to two decimal places, will be closest to:

A 6.54 m **B** 6.68 m **C** 7.00 m **D** 7.08 m **E** 7.25 m

[VCAA 2010]

Extended-response questions

1 A service-station storage tank needs refilling as there are only 1500 litres left in the tank. Petrol is pumped into the tank at the rate of 750 litres per minute.

a How much petrol is in the tank at the start of the third minute?

b Write down an expression for the amount of petrol in the tank, A_n, at the start of the nth minute.

c The tank holds 15 000 litres. How long does it take to fill the tank?

How to generate a sequence defined by a difference equation using the TI-Nspire CAS

Generate the first five terms of the sequence defined by the difference equation
$t_{n+1} = 3t_n - 1$ where $t_1 = 2$.

Method 1 (using recursion)

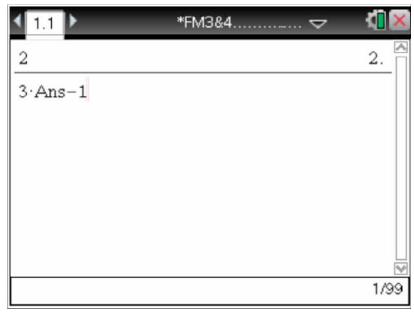
1 Write down the rule for the difference equation and the value of the first term.

$t_{n+1} = 3t_n - 1$ where $t_1 = 2$

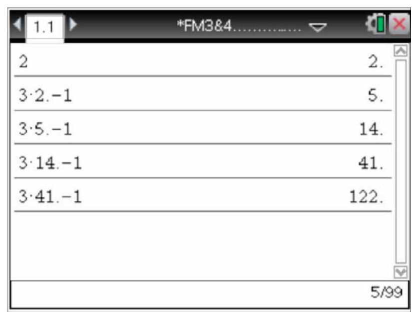
2 Start a new document by pressing (ctrl) + **N**. Select **Add Calculator**.

3 Type **2**, the value of the first term. Press (enter). The calculator stores the value 2 as **Ans**wer. (You can't see this yet.)

4 Now type **3**× **Ans** − **1**, using the keystrokes (3) (x) (ctrl) (−) (−) (1), then press (enter). The second term in the sequence is 5. This value is now stored as **Ans**.



5 Press (enter) to generate the next term. Continue pressing (enter) until the required number of terms is generated.



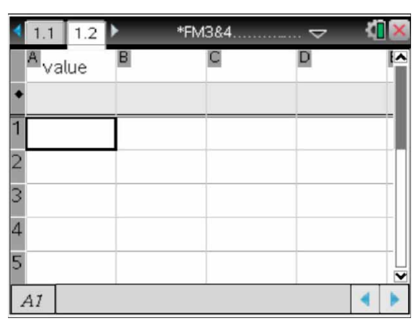
Method 2 (using Sequence Mode)

1 Write down the rules for the difference equation and the value of the first term.

$t_{n+1} = 3t_n - 1$ where $t_1 = 2$

2 Open a **Lists and Spreadsheet** application page and name column A *value*.

Note: We are naming the column now because we want to graph the sequence later.



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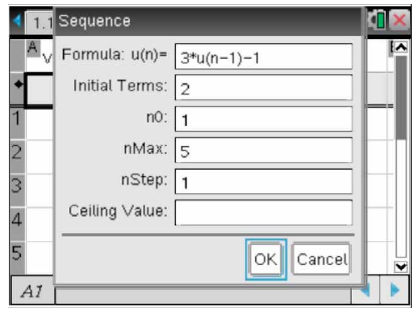
3 With the cursor in column A, press (menu) **>Data>Generate Sequence** to display the pop-up screen shown opposite.
Type in the entries as shown. Use (tab) to move between entry boxes. The **Ceiling Value** (i.e. highest value) can be left blank.

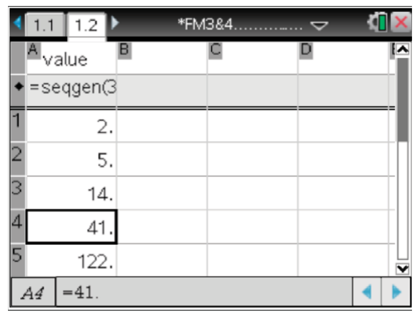


Notes:

1 The calculator uses u $(n-1)$ and u(n) to represent successive terms in the sequence.

2 $t_{n+1} = 3t_n - 1$ can also be written as $t_n = 3t_{n-1} - 1$; hence, we use $u\,(n) = 3 \times u\,(n-1) - 1$ on the calculator.

4 Press (enter) to close the pop-up screen and list the first five terms of the sequence.



Notes:

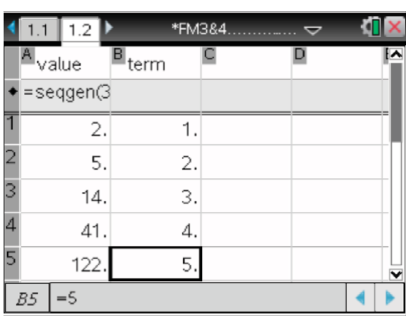
1 The ▼ arrow can be used to move down through the sequence.

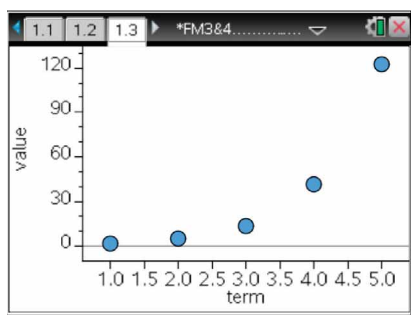
2 The term number can be read directly from the row number (left-hand side) of the spreadsheet. For example, the 4th term in this sequence is 41.

5 The sequence is displayed best using a scatterplot.

a As we are plotting the first five terms only, enter the numbers **1** to **5** in the list called *term* (below left).

b Construct a scatterplot with *term* on the horizontal axis and *value* on the vertical axis to obtain the plot shown below right.





Note: Sequence plotting can also be done in the **Graphs** application.

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Exercise 10F

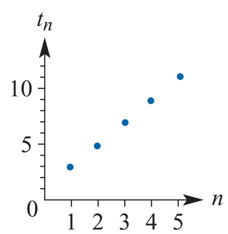
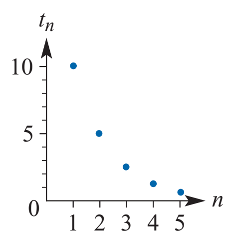
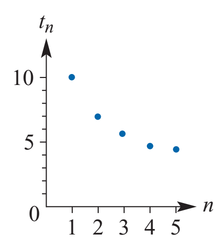
1 The following are general first-order difference equations. Write down their solutions.

a $t_{n+1} = 1.5t_n - 8,\ t_1 = 32$ **b** $t_{n+1} = 0.5t_n + 14,\ t_1 = 20$

c $t_{n+1} = 0.5t_n - 10,\ t_1 = 20$

10.7 Summary of first-order difference equations

An important skill is the ability to differentiate between the three types of first-order difference equations by the sequences they generate. The table below should help you in this task.

Difference equation	$t_{n+1} = t_n + d,\ t_1 = a$	$t_{n+1} = rt_n,\ t_1 = a$	$t_{n+1} = rt_n + d,\ t_1 = a$
Sequence type	arithmetic	geometric	neither arithmetic nor geometric
***n*th term**	$t_n = a + (n-1)d$	$t_n = ar^{n-1}$	$t_n = ar^{n-1} + d\dfrac{(r^{n-1}-1)}{r-1}$
Example			
equation	$t_{n+1} = t_n + 2,\ t_1 = 3$	$t_{n+1} = 0.5t_n,\ t_1 = 10$	$t_{n+1} = 0.5t_n + 2,\ t_1 = 10$
sequence	3, 5, 7, . . .	10, 5, 2.5, . . .	10, 7, 5.5, . . .
graph			
***n*th term**	$t_n = 3 + (n-1)2$ $= 2n + 1$	$t_n = 10(0.5)^{n-1}$	$t_n = 6 \times 0.5^{n-1} + 4$

Exercise 10G

1 For each of the following first-order difference equations:

i identify as generating an arithmetic sequence, a geometric sequence or neither an arithmetic nor a geometric sequence

ii write the first five terms of the sequences they define

iii plot a graph of t_n against n for $1 \le n \le 5$

iv if *arithmetic or geometric*, write an expression for t_n in terms of n

a $t_{n+1} = t_n - 5$ where $t_1 = 35$
b $t_{n+1} = 0.25t_n$ where $t_1 = 64$
c $t_{n+1} = t_n + 5$ where $t_1 = 0$
d $t_{n+1} = 1.1t_n$ where $t_1 = 100$
e $t_{n+1} = t_n + 0.5$ where $t_1 = 0$
f $t_{n+1} = 1.5t_n - 8$ where $t_1 = 32$
g $t_{n+1} = 0.5t_n + 10$ where $t_1 = 20$
h $t_{n+1} = 0.5t_n + 14$ where $t_1 = 20$
i $t_{n+1} = 0.5t_n - 10$ where $t_1 = 20$
j $t_{n+1} = 0.5t_n + 0.5$ where $t_1 = 1$
k $t_{n+1} = 10t_n$ where $t_1 = 0.1$
l $t_{n+1} = 2t_n - 2$ where $t_1 = 1$

10.8 Applications of first-order difference equations

First-order difference equations are very useful mathematical tools for describing growth and decay in such things as animal populations, investments and loans, and the values of goods and services. You will meet some of these applications in this section.

First-order difference equations can be solved by finding the value of the *n*th term. However, in application problems it is far more convenient and acceptable to use your calculator to generate and manipulate the required terms in the sequences.

Example 8 **Applications: managing a budget**

Jarrad has moved to an interstate university to study law. Over the summer break he has accumulated \$3635. He wants to use the money to pay his living expenses while studying. He plans to allow himself \$165 per week to spend on general living expenses. Assume that Jarrad sticks to his plan.

a Write down a difference equation to describe the reduction in Jarrad's savings week by week.
b Determine the number of weeks his money will last.

Solution

a Write down a difference equation to describe the reduction in Jarrad's savings week by week.

1 Identify, name and define the key variable.

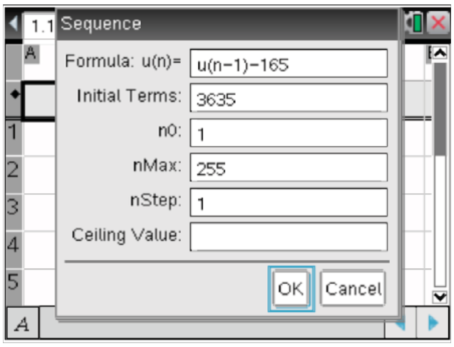
Let S_n be the value of Jarrad's savings at the start of week n.

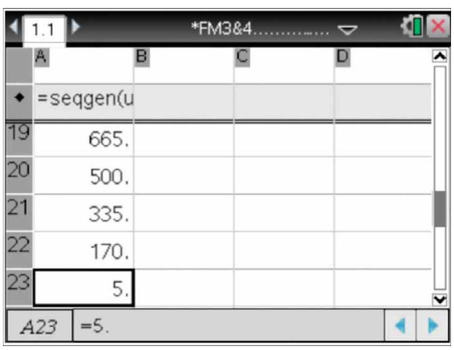
2 Jarrad plans to spend \$165 per week. At the start of week 1 he has \$3635. The sequence is arithmetic. Use this information to write down a difference equation.

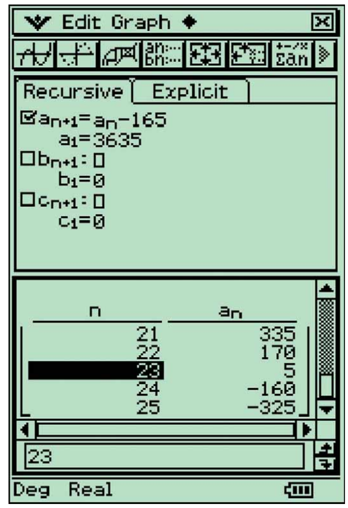
arithmetic sequence:
$d = -165, \ a = 3635$
$\therefore S_{n+1} = S_n - 165$ where $S_1 = 3635$

b Determine the number of weeks his money will last.

1. Use your calculator to list the sequence of terms generated by this difference equation. Find the first term in the sequence that has a value less than 165. This is the 23rd term, which has a value of 5, meaning that Jarrad has only \$5 left to spend at the start of the 23rd week. Thus, Jarrad can only spend \$165 per week for only 22 weeks.







2. Use the down arrow (▾) to find when the nth term of the sequence first has a value below 165. This is week 23, when there is only \$5 left to spend. So Jarrad can spend \$165 per week for only 22 weeks.
3. Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	. . .	21	22	23 . . .
S_n	3635	. . .	335	170	5 . . .

If Jarrad needs \$165 a week, his money will last 22 weeks.

Example 9 Applications: car depreciation

Your neighbour has just bought a new car for \$29 790. When you look up a motoring magazine on secondhand car prices you find out that this particular model of car loses, on average, 17% of its value each year.

a Write down a difference equation to describe the decreasing value of the car each year.

b What will be the secondhand value of the car after your neighbour has owned it for six years; that is, at the start of the seventh year? Give your answer correct to the nearest dollar.

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Solution

a Write down a difference equation to describe the decreasing value of the car each year.

1 Identify, name and define the key variable.

Let V_n be the value of the car at the start of the nth year

2 Use the fact that the car loses 17% of its value each year to write down an expression for V_{n+1} in terms of V_n.

The value of the car at the start of year 1 is $29 790, so $V_1 = 29\,790$.

Combine this information to write down the difference equation.

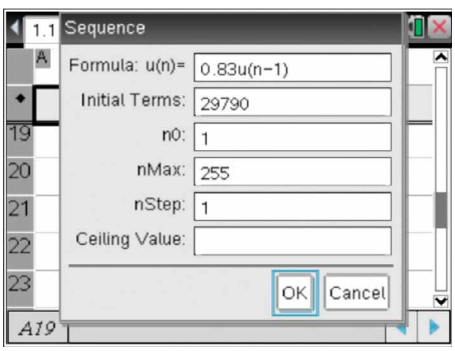
$$V_{n+1} = V_n - 17\% \text{ of } V_n$$
$$= V_n - \frac{17}{100} \times V_n$$
$$= V_n(1 - 0.17) = 0.83V_n$$

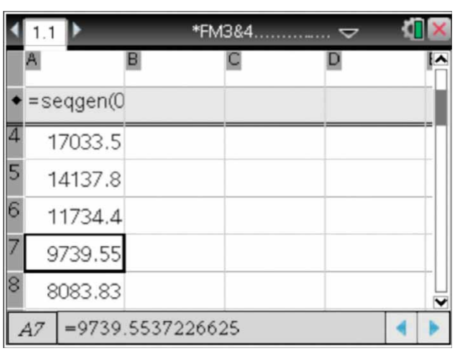
Difference equation:

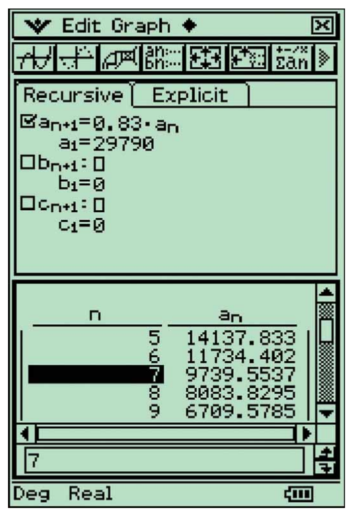
$V_{n+1} = 0.83V_n$ where $V_1 = 29\,790$

b What will be the secondhand value of the car after your neighbour has owned it for six years; that is, at the start of the seventh year? Give your answer correct to the nearest dollar.

1 Use your calculator to list the sequence of terms generated by this difference equation. Find the 7th term.







2 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	...	6	7
V_n	29 790	...	11 734	9739.6

The value of the car after six years is $9740.

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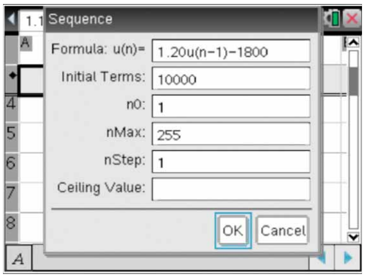
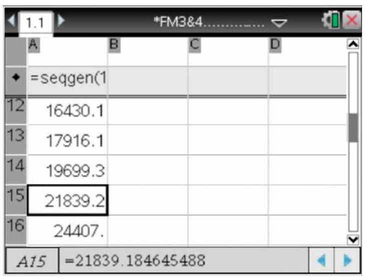
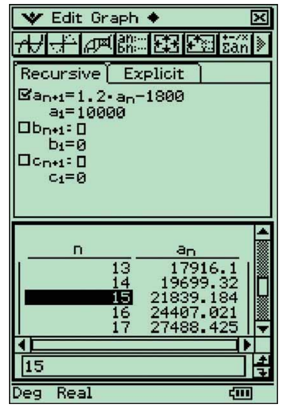
b Under these conditions, how long will it take for trout numbers in the lake to double?

1 Use your calculator to list the sequence of terms generated by this difference equation. Find the first term in the sequence that exceeds 20 000.







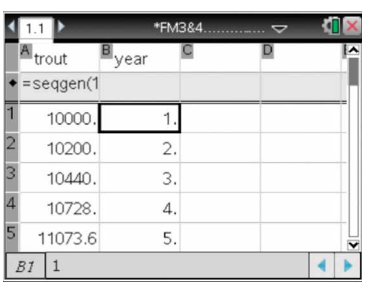
2 Write down key values in the sequence (to show how you solved the problem) and your answer.

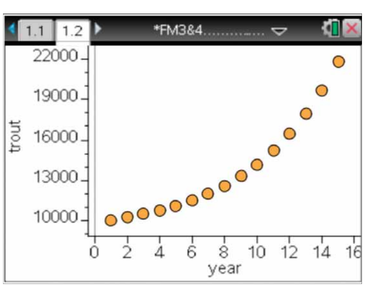
$n =$	1	...	14	15	...
$T_n =$	10 000	...	19 699	21 839	...

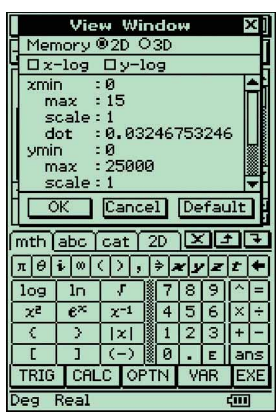
Trout numbers double during the 14 th year.

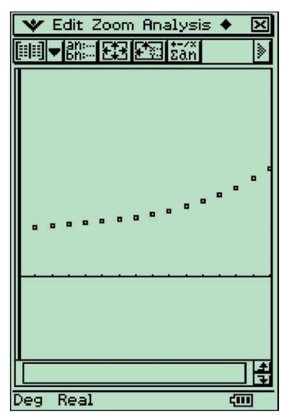
c Graph trout numbers against year for 15 years. Comment on the pattern of growth.

1 Plot the sequence.









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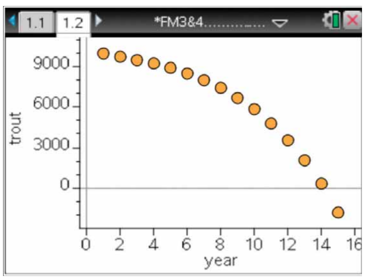
2 Use the plot to comment on the growth in trout numbers. — *Over the 15-year period, trout numbers increase in a non-linear manner.*

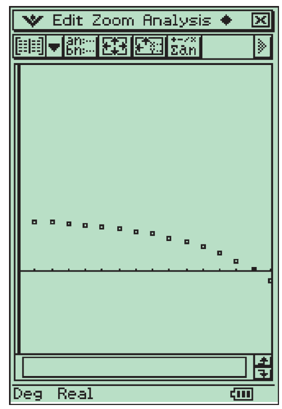
d Suppose that the park authorities had allowed 2200 trout to be fished from the lake each year. Write down a difference equation that determines the number of trout in the lake each year.

1 To investigate the effect of allowing 2200 trout to be fished from the lake, change the 1800 in the original difference equation to 2200. — *Difference equation:* $T_{n+1} = 1.2T_n - 2200,\ T_1 = 10\,000$

e Graph trout numbers against year for 15 years. Comment on the pattern of growth.

1 Replace the value of 1800 in the original difference equation stored in your calculator with 2200 and replot the sequence.

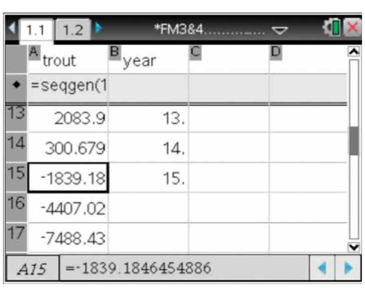


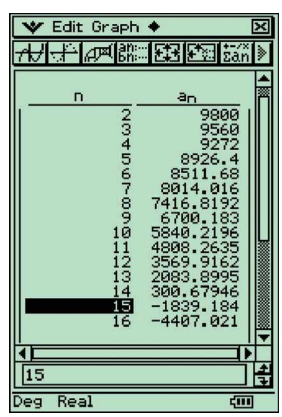


2 Use the plot to comment on the growth in trout numbers. — *If the number of trout that can be fished from the lake is increased to 2200 per year, trout numbers will decrease.*

f If 2200 trout are fished from the lake each year, the trout will disappear. In which year?

1 List the terms of the sequence on your calculator and determine the value of n when a term first becomes zero or negative.





2 Write down key values in the sequence (to show how you solved the problem) and your answer.

n	1	...	14	15	...
Tn	10 000	...	300.68	−1839	...

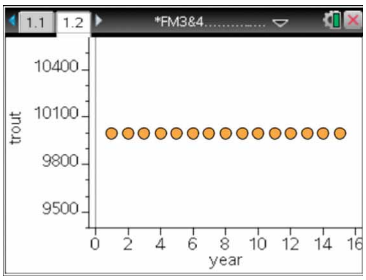
The trout will disappear from the lake during the 14th year.

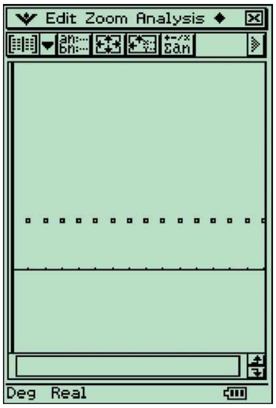
g Suppose that the park authorities allow 2000 trout to be fished from the lake each year. Investigate.

1 The effect of allowing 2000 fish to be taken from the lake by anglers can be investigated by changing the 2200 in the original difference equation to 2000.

Difference equation:
$T_{n+1} = 1.2T_n - 2200,\ T_1 = 10\,000$

Replace the value of 2200 in the original difference equation stored in your calculator with 2000 and replot the sequence.





2 Use the plot to comment on the growth in trout numbers.

Trout numbers will remain constant.

Exercise 10H

1 Rob is offered a job at \$275 per week with yearly increments of \$45 per week.

a Write down a difference equation of the form $W_{n+1} = W_n + d$ where $W_1 = a$ that can be used to describe the growth in Rob's weekly wage year by year. (W_n represents Rob's wage at the start of the nth year.)

b Solve the difference equation by finding an expression for W_n, Rob's wage in year n.

c Use this expression to determine Rob's wage in year 7.

2 Sarah is saving to go on a long overseas holiday. With savings, combined with money earned working as a waitress, she has accumulated \$4670. She wants to use the money to pay her

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Example 11 Generating terms in a Fibonacci sequence from adjacent terms

For the Fibonacci sequence, $t_{11} = 89$, $t_{13} = 233$ and $t_{14} = 377$:

a Determine the values of: **i** t_{15} **ii** $t_9 + t_{10}$ **iii** t_{12}

b Name the terms represented by: **i** $t_{18} + t_{19}$ **ii** $t_{31} - t_{29}$ **iii** $t_{20} + t_{21} + t_{23} + t_{24} - t_{22}$

Solution

Strategy: The key fact in answering all of these questions is that $t_n = t_{n-2} + t_{n-1}$ for $n > 2$. Or, in words, 'after the first two terms, each successive term is the sum of the preceding two terms'.

$t_{11} = 89,\ t_{13} = 233,\ t_{14} = 377$

a **i** $t_{15} = t_{13} + t_{14} = 233 + 377 = 610$

ii $t_9 + t_{10} = t_{11} = 89$

iii $t_{11} + t_{12} = t_{13}$

$\therefore t_{12} = t_{13} - t_{11} = 233 - 89 = 144$

b **i** $t_{18} + t_{19} = t_{20}$

ii $t_{31} - t_{29} = (t_{30} + t_{29}) - t_{29} = t_{30}$

iii
$$\begin{aligned} t_{20} + t_{21} + t_{23} + t_{24} - t_{22} &= (t_{20} + t_{21}) + (t_{23} + t_{24}) - t_{22} \\ &= t_{22} + t_{25} - t_{22} \\ &= t_{25} \end{aligned}$$

When wanting to generate and/or graph more than just a few terms of the Fibonacci sequence from its difference equation, the usual procedure is to use a calculator.

How to generate and group the terms of the Fibonacci sequence using the TI-Nspire CAS

Generate the terms of the Fibonacci sequence given the difference equation:

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.

Graph the first 10 terms.

Steps

1 Write down the rule and the values of the first two terms.

$t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1, t_2 = 1$.

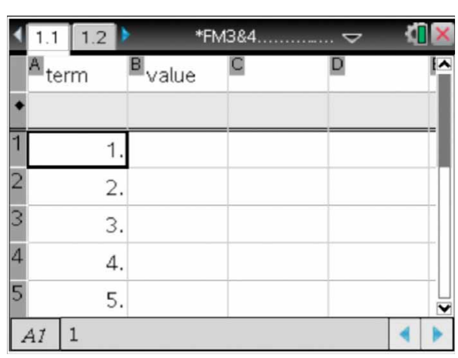
2 Start a new document by pressing ctrl + N.

   a Select **Add Lists & Spreadsheet**.

   b Enter the data **1–10** into a list named *term*, as shown. This is needed later when we come to plot the sequence.

   Note: You can also use the sequence command to do this.

   c Name the list *value* in column B. We will use this column to list the terms of the sequence.



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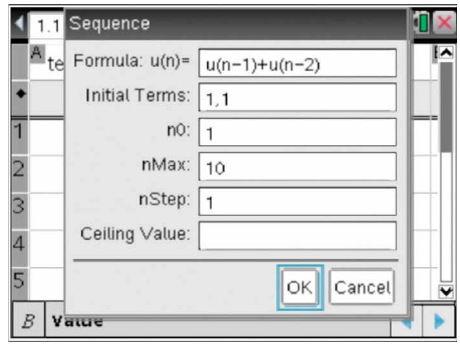
d Place the cursor in any cell in column B and press (menu) **>Data>Generate Sequence**. This will generate the pop-up screen opposite.



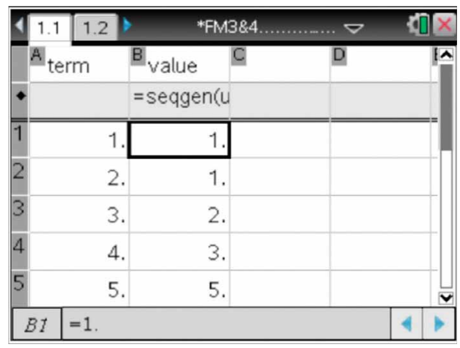
e On this screen, type in the entries as shown. Use (tab) to move between entry boxes.

Notes:

1 Second-order difference equations need two initial terms to be specified.

2 Leave the **Ceiling Value** box blank.

f Press (enter) to list the sequence of terms, as shown.

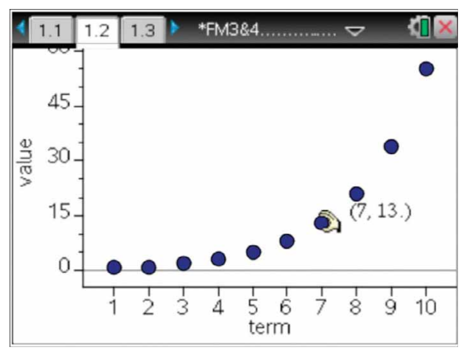


3 Graph the sequence by constructing a scatterplot using *term* as the independent variable and *value* as the dependent variable.

Notes:

1 You can read the values from the graph by placing the cursor on the data point. Alternatively use (menu) **>Analyze>Graph Trace**.

2 Sequence graphing can also be done in the **Graphs** application.



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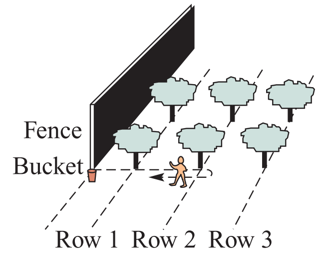
row further each time and bringing back one apple, until the bucket contains 10 apples.

Calculate the shortest distance a child can run to complete this game.



[based on VCAA pre 2006]

6 The charge, in dollars, for a single trip on a toll way depends on the number of sections of road that a motorist travels and the type of toll pass that the motorist uses.

a Using toll pass A, the charge for travelling along n sections of road in a single trip on the toll way is given by the nth term of the arithmetic sequence: \$4.50, \$6.20, \$7.90, . . .

i Show that the common difference for this sequence is \$1.70.

ii Find the charge for travelling along five sections of road in a single trip on the toll way using toll pass A.

iii One motorist paid \$16.40 for a single trip on the toll way using toll pass A. How many sections of road did this motorist travel along?

iv At one entry point, 15 motorists entered the toll way. The first motorist travelled along one section of road. The second motorist travelled along two sections of road. The third motorist travelled along three sections of road and so on. Find the total amount of money that these 15 motorists paid for their trips, assuming that they all used toll pass A.

v Using toll pass A, the charge, in dollars, A_n, for travelling along n sections of road in a single trip on the toll way is given by the difference equation:

$$A_{n+1} = mA_n + k \qquad A_1 = 4.50.$$

Write the values of m and k.

b Different charges apply when motorists use toll pass B. With toll pass B, the charge, in dollars, B_n, for travelling along n sections of road in a single trip on the toll way is given by the difference equation:

$$B_{n+1} = 0.9B_n + 3 \qquad B_1 = 5$$

i Explain the meaning of $B_1 = 5$ in terms of the context of this problem.

ii Find B_3, the charge for travelling along three sections of road in a single trip using toll pass B.

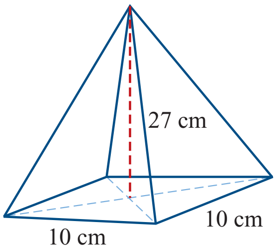
iii This difference equation indicates that there is a maximum charge that motorists who use toll pass B may pay. What is this maximum charge?

c A motorist wishes to get the best value for money when travelling on the toll way. Compare the charges for a single trip using toll pass A and toll pass B. Explain when it would be better for the motorist to use each pass.

[VCAA 2010]

Example 15 **Volume of a pyramid**

Find the volume of this square pyramid with a square base with each edge 10 cm and a height of 27 cm.



Solution

$$V = \frac{1}{3}x^2 h$$
$$= \frac{1}{3} \times 10 \times 10 \times 27$$
$$= 900 \text{ cm}^3$$

Volume of a cone

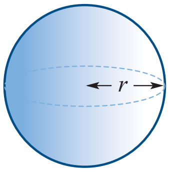
The formula for finding the volume of a cone can be stated as:

Volume of cone $= \frac{1}{3} \times$ base area $\times$ height

$$V = \frac{1}{3}\pi r^2 h$$

Volume and surface area of a sphere

The formulas for the volume and the surface area of a sphere are:



$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

where r is the radius of the sphere.

Example 16 **Volume of a sphere and a cone**

Find the volume of a sphere with radius 4 cm and a cone with radius 4 cm and height 10 cm.

Solution

Volume of sphere $= \frac{4}{3}\pi r^3$
$= \frac{4}{3} \times \pi \times 4^3$
$= 268.08 \text{ cm}^3$ (two d.p.)

Volume of cone $= \frac{1}{3}\pi r^2 h$
$= \frac{1}{3} \times \pi \times 4^2 \times 10$
$= 167.55 \text{ cm}^3$ (two d.p.)

Example 17 **Surface area of a sphere**

Find the surface area of a sphere with radius 10 cm.

Solution

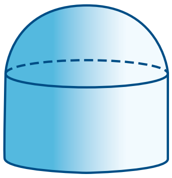
Surface area of sphere $= 4\pi r^2$
$= 4\pi \times 10^2$
$= 1256.64 \text{ cm}^2$ (2 d.p.)

Composite shapes

Using the shapes above, composite shapes can be made. The volumes of these can be found by summing the volumes of the component shapes.

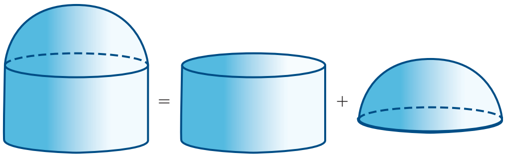
Example 18 Volume of a composite shape

A hemisphere is placed on top of a cylinder to form a capsule. The radius of both the hemisphere and the cylinder is 5 mm. The height of the cylinder is also 5 mm. What is the volume of the composite solid in cubic millimeters, correct to two decimal places?



Solution

1 The composite shape is made up from a cylindrical base plus a hemispherical top. The volume of the composite shape is the volume of the cylinder plus the volume of the hemisphere (half a sphere)



2 Use the formula $V = \pi r^2 h$ to find the volume of the cylinder.

The volume of the cylinder
$V_{cyl.} = \pi \times 5^2 \times 5 = 392.699\ldots\ mm^3$

3 Find the volume of hemisphere noting that the volume of a hemisphere is $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$.

The volume of the hemisphere
$V_{hem.} = \frac{1}{2} \times \frac{4}{3}\pi \times 5^3 = 261.799\ldots\ mm^3$

4 Add the two together.

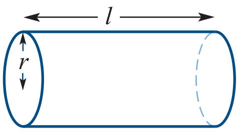
The volume of the composite $= 654.50\ mm^3$

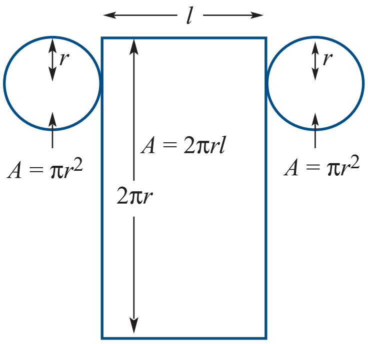
5 Write down your answer.

(correct to two decimal places)

Surface area of three-dimensional shapes

The surface area of a solid can be found by calculating and totalling the area of each of its surfaces. The **net** of the cylinder in the diagram demonstrates how this can be done.





The surface area of the cylinder

$= $ area of ends $+$ area of curved surface

$= $ area of two circles $+$ area of rectangle

$= 2 \times \pi r^2 + 2\pi r \times l = 2\pi r^2 + 2\pi rl$

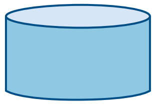
The formulas for the surface areas of some common three-dimensional shapes follow.

Skillsheet TEACHER

Exercise 12F

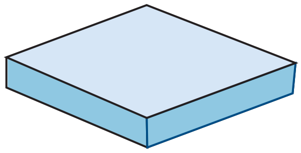
1 Find the volume in cm^3 of each of the following shapes, correct to two decimal places.

a



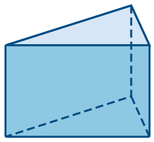
radius 6.3 cm and height 2.1 cm

b



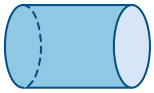
dimensions 2.1 cm, 8.3 cm and 12.2 cm

c



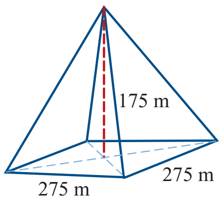
area of cross section $= 2.8$ cm^2
height $= 6.2$ cm

d



radius 2.3 cm and length 4.8 cm

2 Each side of the square base of one of the great Egyptian pyramids is 275 m long. It has a perpendicular height of 175 m. Calculate the volume of this pyramid, correct to the nearest cubic metre.

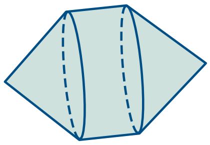


3 Find the volume, correct to one decimal place, of a:

a sphere with radius 1.5 m
b cone with radius 6 cm and height 15 cm
c hemisphere of diameter 3.8 mm
d cone with diameter 15 mm and height 10 mm

4 The diagram shows a capsule, which consists of two hemispheres, each of radius 2 cm, and a cylinder of length 5 cm and radius 2 cm. Find the volume of the capsule correct to the nearest cm^3.

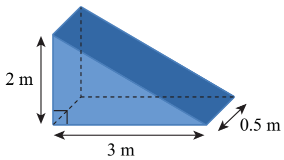
5 The diagram shows a composite shape made from a cylinder and two cones. Both the cylinder and the two cones have a radius of 12 cm. The length of the cylinder is 8 cm and height of the cones is 10 cm. Find the volume of the composite shape. Give your answer correct to the nearest cm^3.



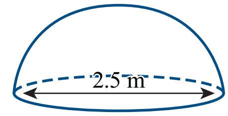
6 Find the total surface areas of shapes **a** and **b** of question **1**. Give answers correct to the nearest cm^2.

7 For the triangular prism shown, find

a the volume in m^3
b the *total* surface area in m^2, correct to one decimal place



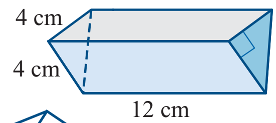
8 A hemispherical dome tent has a diameter of 2.5 m, as shown.



a Determine the volume enclosed by the tent, correct to the nearest m^3.

b Determine the total surface area of the tent (including its floor), correct to the nearest m^2.

9 Find, correct to two decimal places, the surface area and volume of the solid shown given that the cross-section is a right-angled isosceles triangle.

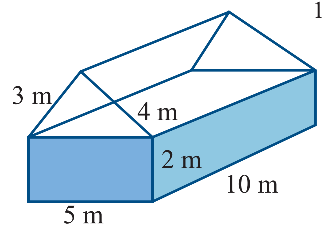


10 Find:

a the surface area

b the volume

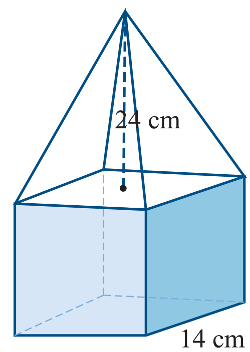
of the object shown.



11 The diagram opposite shows a right pyramid on a cube.
Each edge of the cube is 14 cm.
The height of the pyramid is 24 cm.
Find:

a the volume of the solid

b the surface area of the solid

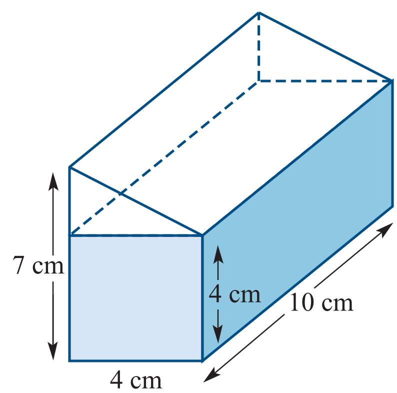


12 Find:

a the surface

b the volume

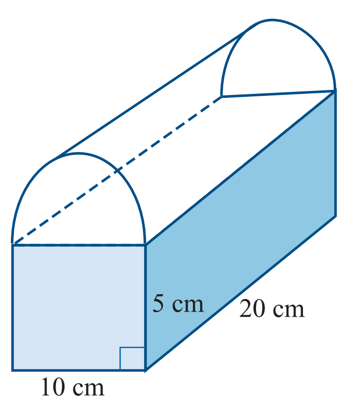
of the solid shown opposite.



13 The solid opposite consists of a half cylinder on a rectangular prism. Find, correct to two decimal places:

a the surface area

b the volume

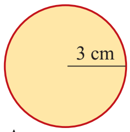


12.7 Areas, volumes and similarity

Areas

Some examples of similar shapes and the ratio of their areas are considered in the following.

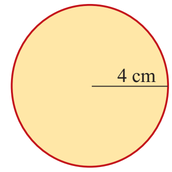
Similar circles



Area $= \pi \times 3^2$

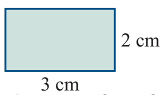
$$\text{Scale factor} = k = \frac{\text{radius circle 2}}{\text{radius circle 1}} = \frac{4}{3}$$

$$\textbf{Ratio of areas} = \frac{\pi \times 4^2}{\pi \times 3^2} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2 = k^2$$



Area $= \pi \times 4^2$

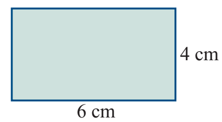
Similar rectangles



Area $= 3 \times 2$
$= 6 \text{ cm}^2$

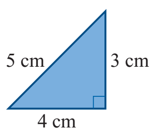
$$\text{Scale factor} = k = \frac{\text{length rectangle 2}}{\text{length rectangle 1}} = \frac{6}{3} = 2$$

$$\textbf{Ratio of areas} = \frac{24}{6} = 4 = (2)^2 = k^2$$



Area $= 6 \times 4$
$= 24 \text{ cm}^2$

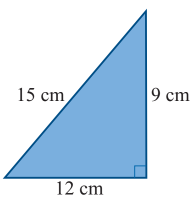
Similar triangles



Area $= \frac{1}{2} \times 4 \times 3$
$= 6 \text{ cm}^2$

$$\text{Scale factor} = k = \frac{\text{height triangle 2}}{\text{height triangle 1}} = \frac{9}{3} = 3$$

$$\textbf{Ratio of areas} = \frac{54}{6} = 9 = (3)^2 = k^2$$



Area $= \frac{1}{2} \times 12 \times 9$
$= 54 \text{ cm}^2$

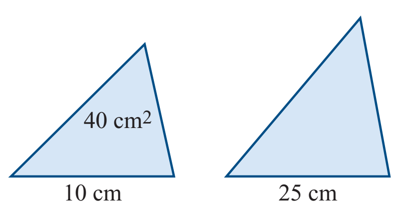
A similar pattern emerges for other shapes. Scaling the linear dimension of a shape by a factor of k scales the area by a factor of k^2.

Scaling areas

If two **shapes are similar** and the **scale factor is *k***, then the **area** of the similar shape $= \boldsymbol{k^2} \times$ area of the original shape.

Example 20 **Using area scale factors with similarity**

The two triangles shown are similar.
The base of the smaller triangle has a length of 10 cm.
Its area is 40 cm^2.
The base of the larger triangle has a length of 25 cm.
Determine its area.



Solution

1 Determine the scale factor k. $\quad k = \frac{25}{10} = 2.5$

2 Write down the area of the small triangle. $\quad$ Area of small triangle $= 40$ cm^2

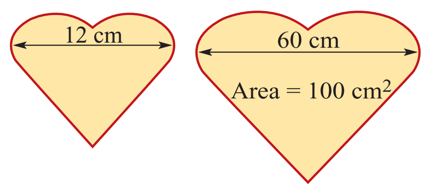
3 Area of larger triangle $= k^2 \times$ area of smaller triangle.
Substitute the appropriate values and evaluate.

$$\therefore \text{Area of larger triangle} = 2.5^2 \times 40 = 250$$

4 Write down your answer. $\quad$ The area of the larger triangle is 250 cm^2.

Example 21 Scale factors and area

The two hearts shown are similar shapes.
The width of the larger heart is 60 cm.
Its area is 100 cm^2.
The width of the smaller heart is 12 cm.
Determine its area.



Solution

1 Determine the scale factor k. Note we are scaling down. $\quad k = \frac{12}{60} = 0.2$

2 Write down the area of the larger heart. $\quad$ Area of larger heart $= 100$ cm^2

3 Area of smaller heart $= k^2 \times$ area of larger heart.
Substitute the appropriate values and evaluate.

$$\therefore \text{Area of smaller heart} = 0.2^2 \times 100 = 4$$

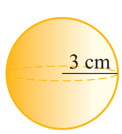
4 Write down your answer. $\quad$ The area of the smaller heart is 4 cm^2.

Volumes

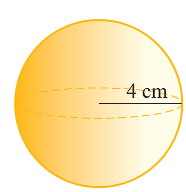
Two solids are considered to be similar if they have the same shape and the ratio of their corresponding linear dimensions is equal.

Some examples of similar volume and the ratio of their areas are considered in the following.

Similar spheres



$$\text{Scale factor} = k = \frac{\text{radius sphere 2}}{\text{radius sphere 1}} = \frac{4}{3}$$

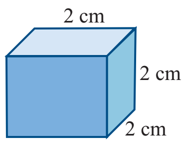


Volume $= \frac{4}{3}\pi \times 3^3 = 36\pi$ cm^3

$$\textbf{Ratio of volumes} = \frac{\frac{256}{3}\pi}{36\pi} = \frac{256}{108} = \frac{64}{27} = \left(\frac{4}{3}\right)^3 = k^3$$

Volume $= \frac{4}{3}\pi \times 4^3 = \frac{256}{3}\pi$ cm^3

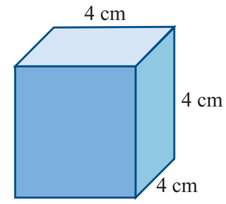
Similar cubes



Volume $= 2 \times 2 \times 2$
$= 8 \text{ cm}^3$

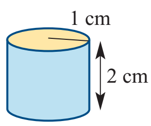
Scale factor $= k = \dfrac{\text{side length 2}}{\text{side length 1}} = \dfrac{4}{2} = 2$

Ratio of volumes $= \dfrac{64}{8} = 8$
$= (2)^3 = k^3$



Volume $= 4 \times 4 \times 4$
$= 64 \text{ cm}^3$

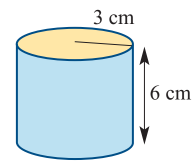
Similar cylinders



Volume $= \pi \times 1^2 \times 2$
$= 2\pi \text{ cm}^3$

Scale factor $= k = \dfrac{\text{radius 2}}{\text{radius 1}} = \dfrac{3}{1} = 3$

Ratio of volumes $= \dfrac{54\pi}{2\pi} = 27 = (3)^3 = k^3$



Volume $= \pi \times 3^2 \times 6$
$= 54\pi \text{ cm}^3$

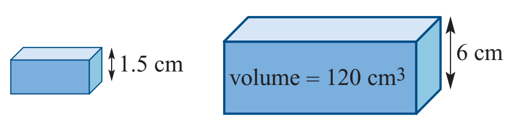
A similar pattern emerges for other solids. Scaling the linear dimension of a solid by a factor of k scales the volume by a factor of k^3.

Scaling volumes

If two **solids are similar** and the **scale factor is** $\boldsymbol{k}$, then the **volume** of the similar solid $= \boldsymbol{k}^3 \times$ volume of the original solid.

Example 22 Similar solids

The two cuboids shown are similar solids.
The height of the larger cuboid is 6 cm.
Its volume is 120 cm^3.
The height of the smaller cuboid is 1.5 cm.
Determine its volume.



Solution

1 Determine the scale factor k. Note that we are scaling down.

$k = \dfrac{1.5}{6} = 0.25$

2 Write down the volume of the larger cuboid.

Volume larger cuboid $= 120 \text{ cm}^3$

3 Volume smaller cuboid $= k^3 \times$ volume larger cuboid.
Substitute the appropriate values and evaluate.

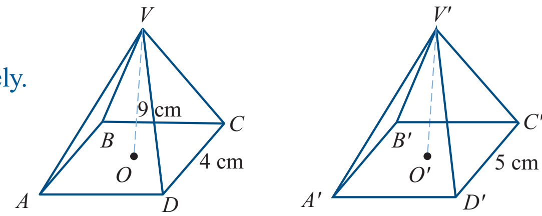
Volume smaller cuboid $= 0.25^3 \times 120$
$= 1.875$

4 Write down your answer.

The volume of the smaller cuboid is 1.875 cm^3.

Example 23 Similar solids

The two square pyramids shown are similar with a base dimensions 4 and 5 cm, respectively. The height of the first pyramid is 9 cm and its volume is 48 cm^3. Find the height and volume of the second pyramid.



Pyramid 1 Pyramid 2

Solution

1 Determine the scale factor, k. Use the base measurements. $k = \dfrac{5}{4} = 1.25$

Height

2 Write down the height of Pyramid 1. Height 1 = 9 cm

3 Height Pyramid 2 = $k \times$ height Pyramid 1. Substitute the appropriate values and evaluate. $\therefore$ Height 2 = $1.25 \times 9 = 11.25$

4 Write down your answer. The height of Pyramid 2 is 11.25 cm.

Volume

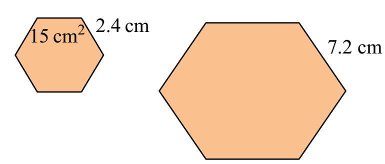
5 Volume Pyramid 2 = $k^3 \times$ volume Pyramid 1. Substitute the appropriate values and evaluate. Volume 1 = 48 cm^3

$\therefore$ Volume 2 = $1.25^3 \times 48 = 93.75$

6 Write down your answer. The volume of Pyramid 2 is 93.75 cm^3.

Exercise 12G

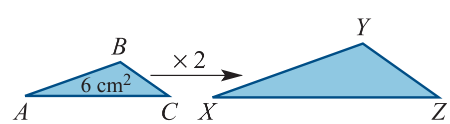
1 Two regular hexagons are shown.
The side length of the smaller hexagon is 2.4 cm.
The side length of the larger hexagon is 7.2 cm.



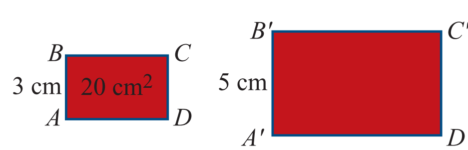
a Determine the length scale factor k for scaling up.

b The area of the smaller hexagon is 15 cm^2. Determine the area of the larger hexagon.

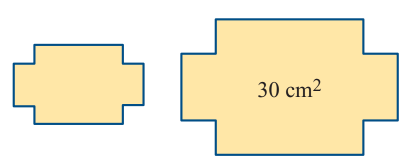
2 Triangle ABC is similar to triangle XYZ. The length scale factor $k = 2$. The area of triangle ABC is 6 cm^2. Find the area of triangle XYZ.



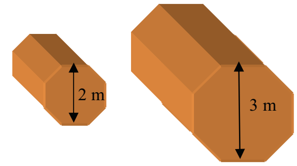
3 The two rectangles are similar. The area of rectangle $ABCD$ is 20 cm^2. Find the area of rectangle $A'B'C'D'$.



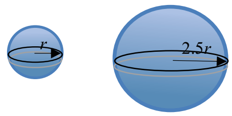
4 The two shapes shown are similar. The length scale factor for scaling down is $\frac{2}{3}$. The area of the shape to the right is 30 cm^2. What is the area of the shape to the left?



5 The octagonal prisms are similar.
The height of the smaller prism is 2 m. The height of the larger prism is 3 m.
The surface area of the smaller prism is 18 m^2.
Determine the surface area of the larger prism in m^2.

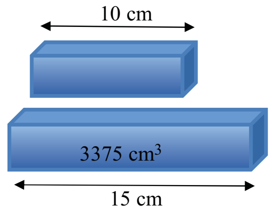


6 The radius of the larger sphere is 2.5 times the radius of the smaller sphere. The volume of the smaller sphere is 24 mm^3.

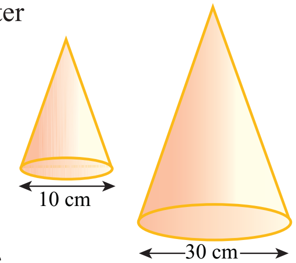


a Write down the length scale factor k for scaling up.
b Determine the volume of the larger sphere in mm^3.

7 The two rectangular prisms are similar.
The length of the smaller prism is 10 cm. The length of the larger prism is 15 cm.
The volume of the larger prism is 3375 cm^3.
Determine the volume of the smaller prism in cm^3.

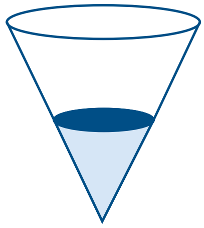


8 The two cones shown are similar. The smaller cone has a diameter of 10 cm. The larger cone has a diameter of 30 cm.

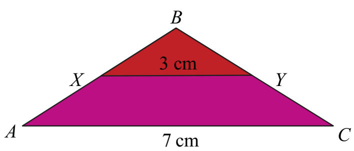


a Determine the length scale factor k for scaling up.
b What is the length scale factor k for scaling down?
c The height of the larger cone is 45 cm. Determine the height of the smaller cone.
d The surface area of the smaller cone is 326.9 cm^2. Determine the surface area of the larger cone correct to the nearest cm^2.
e The volume of the smaller cone is 392.7 cm^3. What is the volume of the larger cone, correct to the nearest cm^3?

9 An inverted right circular cone of capacity 100 m^3 is filled with water to half its depth. Find the volume of water.



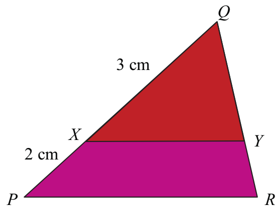
10 Triangles XBY and ABC are similar.
The area of triangle XBY is 1.8 cm^2.
Determine the area of triangle ABC.



MC TEST

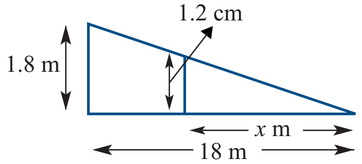
D&D TEST

11 Triangles XQY and PQR are similar.
The area of triangle PQR is 7.5 cm^2.
a Determine k, the length scale factor for scaling down.
b Determine the area of triangle XQY.



11 The value of x is:

A 12 **B** 27 **C** 2.16
D 20.8 **E** 13.81

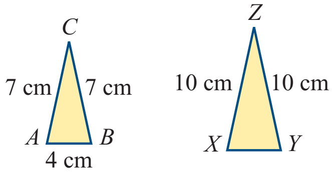


12 A regular convex polygon has 12 sides. The magnitude of each of its interior angles is:

A 30° **B** 45° **C** 60° **D** 150° **E** 120°

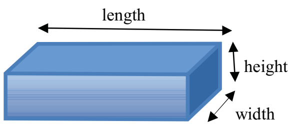
13 Triangles ABC and XYZ are similar isosceles triangles.
The length of XY, correct to one decimal place, is:

A 4.8 cm **B** 5.7 cm **C** 4.2 cm
D 8.5 cm **E** 8.2 cm



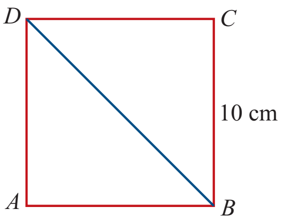
14 The rectangular prism shown has a volume of 12.8 cm^3.
A second rectangular prism is made with half the length, four times the height and double the width.
The volume of the second prism (in cm^3) is:

A 6.4 **B** 12.8 **C** 51.2
D 102.4 **E** 204.8



15 Each side length of a square is 10 cm. The length of the diagonal is:

A 10 **B** $5\sqrt{2}$ **C** $10\sqrt{2}$ **D** 8 **E** 1.4

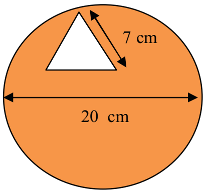


16 To the nearest mm^2, the surface area of a sphere of radius of radius 8 mm is:

A 202 mm^2 **B** 268 mm^2 **C** 804 mm^2
D 808 mm^2 **E** 2145 mm^2

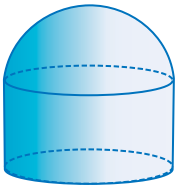
17 An equilateral triangle of side length 7 cm is cut from a circular sheet of metal of diameter 20 cm.
The area of the resulting shape (in cm^2) is closest to:

A 21 **B** 293 **C** 314
D 335 **E** 921



18 The diagram shows a composite shape that consists of a hemisphere of radius 6 cm placed on top of a cylinder of height 8 cm and radius 6 cm.
The total surface area of the composite shape (including the base) is closest to:

A 302 cm^2 **B** 452 cm^2 **C** 528 cm^2
D 641 cm^2 **E** 754 cm^2

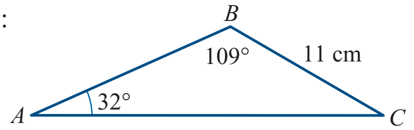


9 The area of the triangle ABC, where $b = 5$ cm, $c = 3$ cm, $\angle A = 30^\circ$ and $\angle B = 70^\circ$, is:

A 2.75 cm^2 **B** 3.75 cm^2 **C** 6.50 cm^2 **D** 7.50 cm^2 **E** 8 cm^2

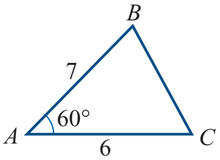
10 The length of AC, correct to one decimal place, is:

A 6.2 cm **B** 16.3 cm **C** 19.6 cm
D 40.4 cm **E** 20.3 cm



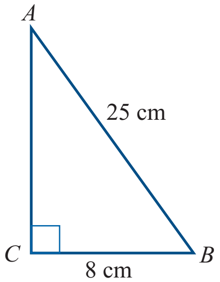
11 The square of the length of side BC is:

A 36 **B** 85 **C** 49
D 42 **E** 43



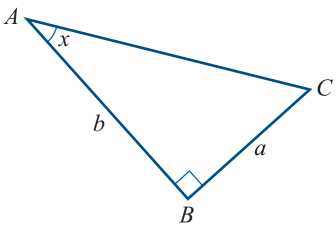
12 For the triangle shown, the value of the cosine of angle ABC is:

A $\frac{8}{25}$ **B** 74 **C** $\frac{5}{6}$
D $\frac{-5}{6}$ **E** 73



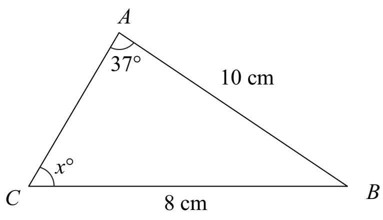
13 In the triangle ABC, $\cos x =$

A $\dfrac{a}{\sqrt{a^2+b^2}}$ **B** $\dfrac{b}{\sqrt{a^2+b^2}}$ **C** $\dfrac{a}{b}$
D $\dfrac{b}{a}$ **E** $\dfrac{\sqrt{a^2+b^2}}{a}$



14 In triangle ABC, $\sin x^\circ =$

A $1.25 \sin(37^\circ)$ **B** $\dfrac{1.25}{\sin(37^\circ)}$ **C** $\dfrac{\sin(37^\circ)}{1.25}$
D $0.8 \sin(37^\circ)$ **E** $\dfrac{0.8}{\sin(37^\circ)}$

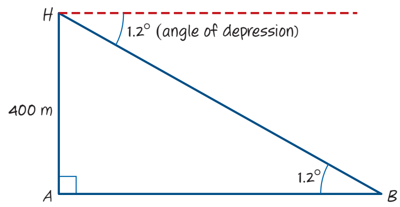


Review

Example 1 **Angle of depression**

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Draw a diagram and calculate the horizontal distance of the boat to the helicopter, correct to the nearest 10 metres.

Solution



$$\frac{AH}{AB} = \tan 1.2°$$

$$\therefore \frac{400}{AB} = \tan 1.2°$$

$$AB = \frac{400}{\tan 1.2}$$

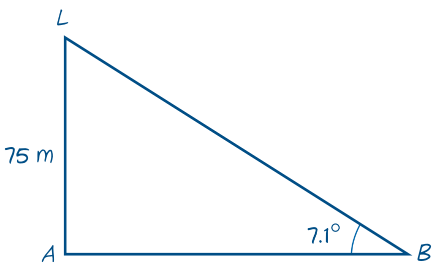
$$AB = 19\,095.80056\ldots$$

The horizontal distance is 19 100 m, to the nearest 10 metres.

Example 2 **Angle of elevation**

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Draw a diagram and calculate the distance of the boat from the lighthouse, to the nearest metre.

Solution



$$\frac{75}{AB} = \tan 7.1°$$

$$\therefore AB = \frac{75}{\tan(7.1°)}$$

$$= 602.135\ \ldots$$

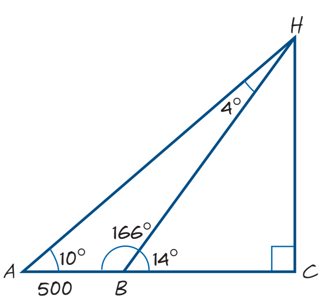
The distance of the boat from the lighthouse is 602 m, to the nearest metre.

Example 3 **Applying geometry and trigonometry with angle of elevation**

From a point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Draw a diagram and find the height of the hill above the level of A, to the nearest metre.

Solution

1 Draw a diagram.



2 Find all the unknown angles that will be required. This is done using properties of angles discussed in Chapter 12.

The magnitude of angle $HBA = (180 - 14)^\circ = 166^\circ$.
The magnitude of angle $AHB = 180 - (166 + 10) = 4^\circ$.

3 You choose to work in particular triangles. In this case it is triangle *ABH*.

Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

$$\therefore HB = \frac{500 \times \sin 10^\circ}{\sin 4^\circ} = 1244.67\ldots$$

4 The information found in triangle *ABH* is the length *HB*. This can now be used to find *HC* in triangle *BCH*.

In triangle BCH:

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\therefore HC = HB \sin 14^\circ = 301.11\ldots$$

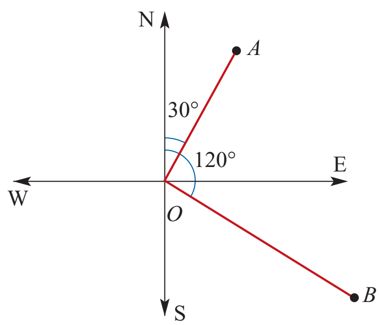
5 Write down your answer.

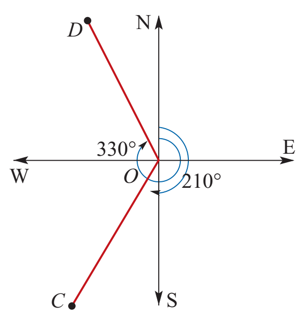
The height of the hill is 301 m, to the nearest metre.

Bearings

The **three-figure bearing** (or compass bearing) is the direction measured clockwise from north.

The bearing of *A* from *O* is 030°.
The bearing of *B* from *O* is 120°.
The bearing of *C* from *O* is 210°.
The bearing of *D* from *O* is 330°.





Example 4 **Bearings and Pythagoras' theorem**

The road from town *A* runs due west for 14 km to town *B*. A television mast is located due south of *B* at a distance of 23 km. Draw a diagram and calculate the distance of the mast from the centre of town *A*, to the nearest kilometre. Find the bearing of the mast from the centre of the town.

Exercise 14A

Angles of depression and elevation

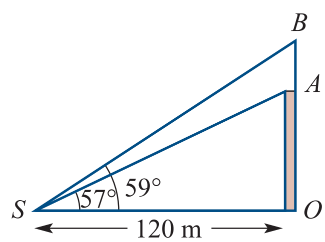
1 The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

2 From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

3 A man standing on top of a mountain observes that the angle of depression to the foot of a building is 41°. If the height of the man above the foot of the building is 500 m, find the horizontal distance from the man to the building.

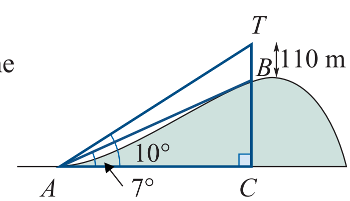
4 A man lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If he is in line with the buoy, calculate the distance between the buoy and the foot of the cliff, which may be assumed to be vertical.

5 Point S is at a distance of 120 m from the base of a building.
On the building is an aerial, AB.
The angle of elevation from S to A is 57°.
The angle of elevation from S to B is 59°.



a Find the distance OA.
b Find the distance OB.
c Find the distance AB.

6 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill the angle of elevation of the bottom of the tower is 7°, and that of the top is 10°.



a Find the magnitude of angles TAB, ABT and ATB.
b Use the sine rule to find the length AB.
c Find CB, the height of the hill.

Bearings

7 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

8 A ship sails 10 km north and then 15 km east. What is its bearing from the starting point?

9 A ship leaves port A and steams 15 km due east. It then turns and steams for 22 km due north.

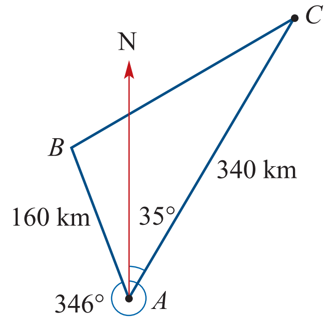
a What is the bearing of the ship from A?
b What is the bearing of port A from the ship?

10 The bearing of a ship, S, from a lighthouse, A, is 055°. A second lighthouse, B, is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°.
The bearing of B from A is 346°.
The distance of C from A is 340 km.
The distance of B from A is 160 km.

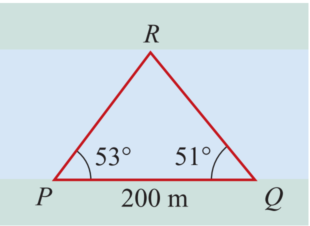
a Find the magnitude of angle BAC.
b Use the cosine rule to find the distance of B to C.



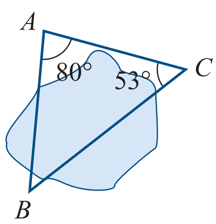
Triangulation

13 P and Q are points on the bank of a river. A tree is at a point, R, on the opposite bank such that $\angle QPR$ is 53° and $\angle RQP$ is 51°.

a Find: **i** RP **ii** RQ
b T is a point between P and Q such that $\angle PTR$ is a right angle. Find RT and hence the width of the river, correct to two decimal places.



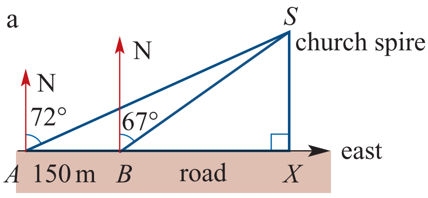
14 Two points, A and B, are on opposite sides of a lake so that the distance between them cannot be measured directly. A third point, C, is chosen at a distance of 300 m from A and with angles BAC and BCA of 80° and 53°, respectively. Calculate the distance between A and B.



Mixed problems

15 A man walking due east along a level road observes a church spire from point A. The bearing of the spire from A is 072°. He then walks 150 m to point B where the bearing is 067°.

a Find the distance of the church spire from B (i.e. BS).
b Find the distance of the church spire from the road (i.e. SX).



16 From a ship, S, two other ships, P, and Q, are on bearings 320° and 075°, respectively. The distance $PS = 7.5$ km and the distance $QS = 5$ km. Find the distance PQ.

17 A yacht starts from point A and sails on a bearing of 035° for 2000 m. It then alters its course to one in a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

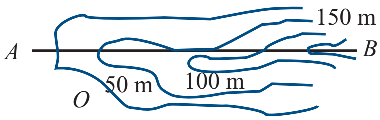
a Find the distance AB. **b** Find the bearing of B from A.

Skillsheet TEACHER

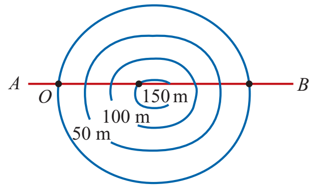
Exercise 14C

1 Draw a cross-sectional profile for each of the following maps with the given cross-section AB.

a



b



2 Two places on a map are 5 cm apart. One is on a 50 m contour and the other on a 450 m contour. If the scale of the map is 1 cm to 1 km, what is the angle of elevation from the first to the second place?

3 a For this diagram the horizontal distance from A to B is 400 m. Find:

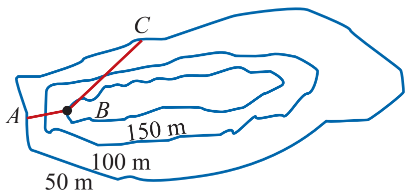
i the distance from A to B

ii the angle of elevation of B from A

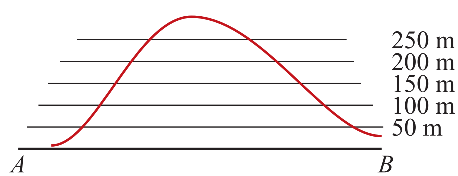
b The horizontal distance from B to C is 1 km. Find:

i the distance of B from C

ii the angle of elevation of B from C

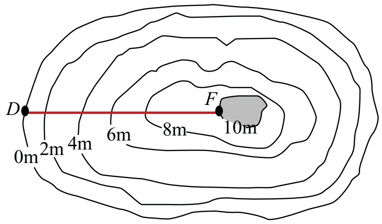


4 Draw a possible contour map to match the given cross-section.



5 A hill with a flat top, shown shaded on the contour map opposite, is reached by climbing a staircase from point D to point F.

On the contour map, 1 cm represents 2 m on the horizontal level.



a The length of the line DF on the contour map is 4.5 cm. What is the horizontal distance (in metres) from D to F?

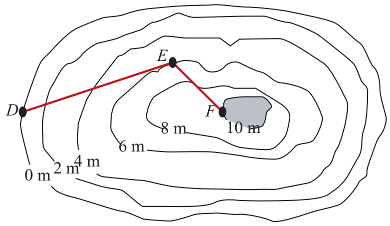
b A staircase is unsafe if its angle of elevation is greater than 45°. Show that the staircase between points D and F on the contour map above would be considered unsafe.

MC TEST

D&D TEST

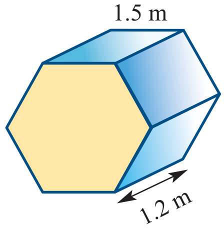
The single staircase between points D and F is replaced by two new staircases, one between D and E, the other between E and F. See the contour map opposite.

c The new staircase from E to F will have a slope of 0.8. Calculate the length of a line drawn on the contour map joining points E and F.



[Based on VCAA 2010]

f The hexagonal blocks are 1.2 m deep. Find the volume of one of the hexagonal blocks. Give your answer in cubic metres, correct to one decimal place.

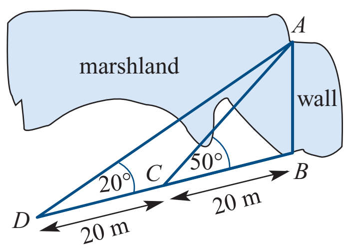


Aristotle wanted to see a scale model of a section of the wall before it was built. The scale he chose was 1 : 25 ($k = \frac{1}{25}$).

g What would be the length of an edge of a hexagonal face of a block for the model? Give your answer in centimetres.

h What is the volume of a block in the model, in cm^3?

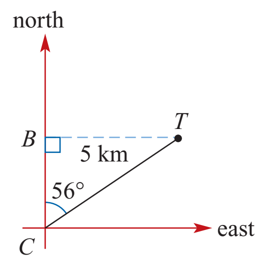
A part of the wall is to cross a marshland. Aristotle wanted to find out the length of this part of the wall but did not want to get his sandals muddy. To overcome the problem, Aristotle made the measurements shown on the diagram.



i Find the distance AC in metres, correct to two decimal places.

j Find the length of the wall to be constructed across the marshland. Give your answer to the nearest metre.

2 From a point C, by looking due north, a girl can see a beacon at point B. She can also see a tower at point T, which is 5 km away on a bearing of 056°. The tower at point T is due east of the beacon at B.

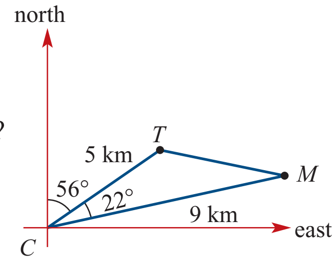


a Calculate the length of BT, the distance of the tower from the beacon. Give your answer correct to three decimal places.

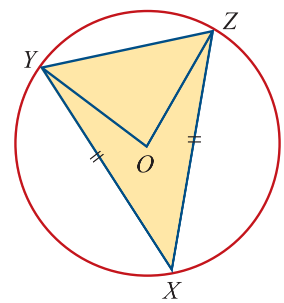
b If she looks a further 22° from the tower at T, the girl can see a radio mast at point M, which is 9 km away.

i What is the bearing of the mast at point M from C?

ii What is the distance between the tower and the mast, correct to three decimal places?



d A mesh is to be placed over the reservoir to partially shade its surface. The first plan is to use a triangular mesh. The triangular mesh, XYZ, is to be supported by three posts around the edge of the reservoir at X, Y and Z, respectively, as shown. In the diagram, $YX = ZX$ and $\angle YOZ$ is a right angle. O is the centre of the circle and $OZ = OY = 50$ m.



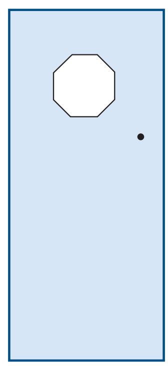
i Find the length YX (correct to two decimal places).

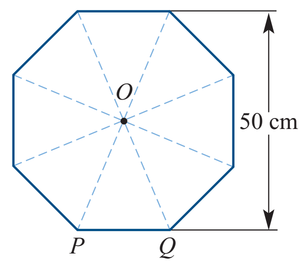
ii Find the area of the triangular mesh (rounded to the nearest whole number).

iii Find the percentage of the area of the circle, centre O, covered by the triangular mesh (correct to one decimal place).

e If the mesh has the form of a regular dodecagon (12-sided regular polygon), with vertices on the circumference of the circle, find the percentage of the area of the circle covered by the mesh (correct to one decimal place).

4 Lee and Nick are staying at the seaside township of Eagle Point, which is famous for the window in its lighthouse door, which is in the shape of a regular octagon. PQ is the bottom side of the window with diagonals that meet at O. The height of the window is 50 cm.

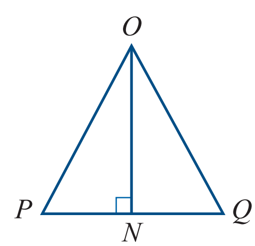

Lighthouse door



Window in door

a Show by calculation that the size of angle POQ is 45°.

b In triangle POQ, N is the midpoint of PQ.



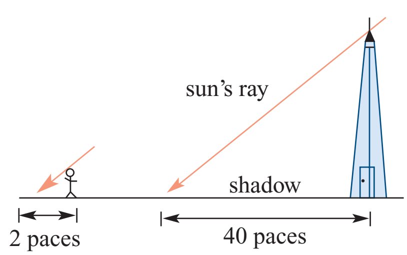
i Write down the length of ON.

ii Write down the size of angle PON.

iii What is the length of PQ in centimetres, correct to two decimal places?

c Find the area of the glass in the octagonal window. Give your answer correct to the nearest square centimetre.

d At midday, the lighthouse casts a shadow directly onto a straight level road leading to the lighthouse. Lee measures the length of the shadow by pacing, and finds that it is 40 paces long when measured from the centre of the base of the lighthouse. When Nick stands on the road, Lee finds that Nick's shadow is two paces long, as shown in the diagram. Nick is 172 cm tall. What is the height of the lighthouse, in metres, correct to one decimal place?

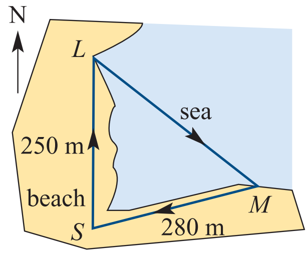


Review

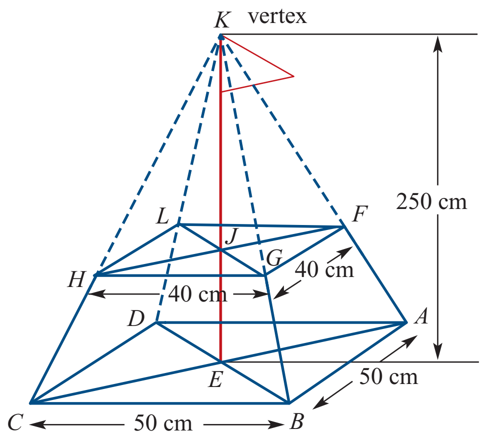
e The Eagle Point Surf Club has set up a training course which requires participants to run 250 metres along the beach from the starting point, S, to a point, L, on the shore. They then swim across an inlet to a point, M, on the opposite shore before running 280 metres directly back to the starting point S, as shown.

L is due north of S and the bearing of M from S is 078°.

- **i** Write down the size of angle LSM.
- **ii** Find the total length of the training course. Give your answer correct to the nearest metre.
- **iii** What is the bearing of M from L? Give your answer correct to the nearest degree.



f The club places flags on the beach to mark points on the training course. The flagpoles sit in wooden boxes, which are in the shape of truncated right pyramids. One such box is shown in the diagram. The base of the box, $ABCD$, is a 50 cm by 50 cm square. The top, $FGHL$, is a 40 cm by 40 cm square. The flagpole, KE, sits vertically in the box and is 250 centimetres long. If the pyramid could be completed, its vertex would be at K, the top of the flagpole, as shown.

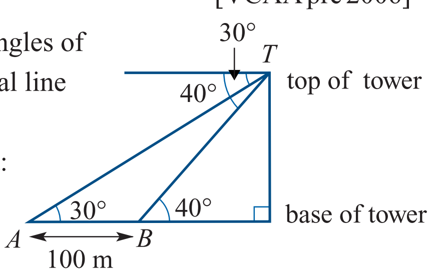


- **i** Find the angle KCE. Give your answer correct to the nearest degree.
- **ii** Find JE, the depth of the block, in centimetres.

[VCAA pre 2006]

5 From the top of a communications tower, the angles of depression of two points A and B on a horizontal line through the foot of the tower are 30° and 40°. The distance between the points is 100 m. Find:

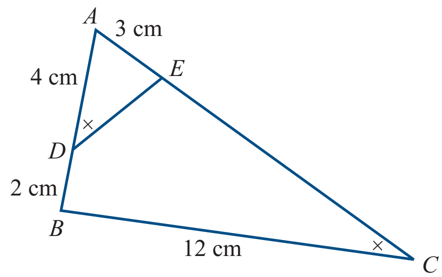
- **a** the distance AT
- **b** the distance BT
- **c** the height of the tower



13 A pencil container is in the shape of a cylinder of length 30 cm and diameter 5 cm. The total surface area of the pencil container (including both ends) is closest to:

A 39 cm^2 **B** 157 cm^2 **C** 471 cm^2 **D** 491 cm^2 **E** 511 cm^2

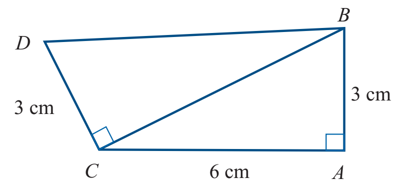
14 D and E are points on AB and AC, respectively. $AD = 4$ cm, $DB = 2$ cm, $AE = 3$ cm and $BC = 12$ cm. The magnitude of $\angle ADE =$ the magnitude of $\angle ACB$. The length DE, in centimetres, is:



A 6 **B** $\frac{9}{2}$ **C** 9

D 10 **E** 11

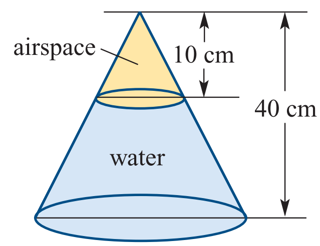
15 In this figure the length of DB, in centimetres, is:



A 6 **B** 9 **C** $3\sqrt{5}$

D $3\sqrt{6}$ **E** $3\sqrt{7}$

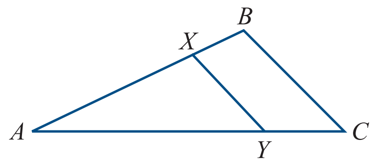
16 A conical container is 40 cm tall and has a capacity of 6032 cm^3. Water is poured into the cone leaving a conical airspace of height 10 cm, as shown in the diagram. The volume of the water, in cm^3, is closest to:



A 94 **B** 1508 **C** 4524

D 5655 **E** 5938

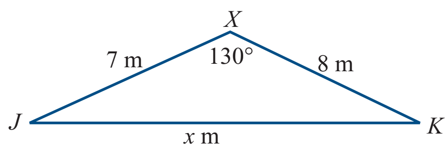
17 $\triangle ABC$ is similar to $\triangle AXY$. $AX = \frac{2}{3}AB$. The area of $\triangle ABC$ is 108 cm^2. The area of $\triangle AXY$ is:



A 32 cm^2 **B** 48 cm^2 **C** 54 cm^2

D 72 cm^2 **E** 81 cm^2

18 Which *one* of the following equations gives the correct value for x?



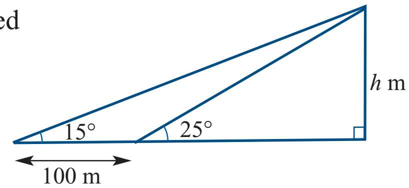
A $x^2 = 49 + 64 + 2\,(7)\,(8)\cos 50^\circ$

B $x^2 = 49 + 64 + 2\,(7)\,(8)\cos 70^\circ$

C $\dfrac{x}{\sin 130^\circ} = \dfrac{8}{\sin 25^\circ}$ **D** $\dfrac{x}{\sin 130^\circ} = \dfrac{7}{\sin 25^\circ}$

E $x^2 = 49 + 64 - 2\,(7)\,(8)\cos 50^\circ$

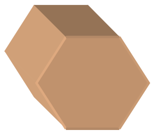
19 The height, h m, of a television tower can be calculated by measuring the angles of elevation of the top of the tower from two points that are in line with the tower but that are 100 m apart. Which *one* of the following equations will give the correct value of h?



A $h = \dfrac{100\sin 15^\circ \tan 25^\circ}{\sin 10^\circ}$ **B** $h = \dfrac{100\sin 15^\circ \sin 25^\circ}{\sin 10^\circ}$ **C** $h = \dfrac{100\sin 10^\circ \tan 25^\circ}{\sin 15^\circ}$

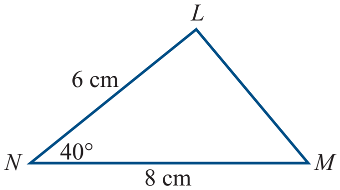
D $h = \dfrac{100\sin 10^\circ \tan 25^\circ}{\sin 15^\circ}$ **E** $h = 100\tan 15^\circ$

32 The hexagonal prism shown has length 2 cm and volume 17.8 cm^3. Its cross-sectional area is closest to:



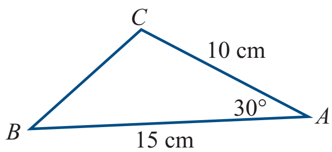
A 2.2 cm^2 **B** 4.5 cm^2 **C** 8.9 cm^2
D 35.6 cm^2 **E** 14.4 cm^2

33 The area of triangle LMN in square centimetres is:



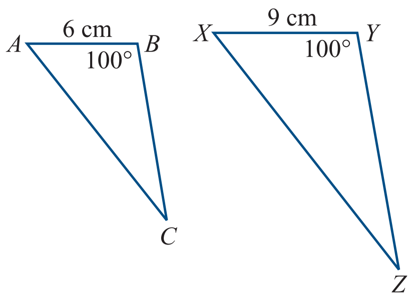
A $\frac{6}{5}\sin 40°$ **B** $\frac{6}{5\cos 40°}$ **C** $15\sin 40°$
D $24\cos 40°$ **E** $24\sin 40°$

34 The area of triangle ABC, in square centimetres, is:



A 15 **B** 37.5 **C** 75
D 90 **E** 150

35 The area of triangle ABC is 20 cm^2. Triangle XYZ is similar to triangle ABC. The area of triangle XYZ, in square centimetres, is:

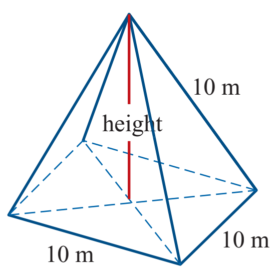


A 30 **B** 35 **C** 40
D 45 **E** 50

36 A vegetable garden has an area of 324 m^2.
When shown on a plan, the vegetable garden has an area of 36 cm^2.
On this plan, 1 cm represents an actual distance of:

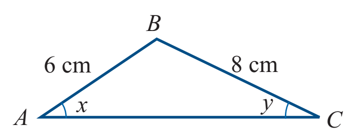
A 1 m **B** 2 m **C** 3 m
D 9 m **E** 18 m

37 A right pyramid with a square base is shown. The square base has sides of length 10 m. The length of each sloping edge is also 10 m. The height of the pyramid in metres is:



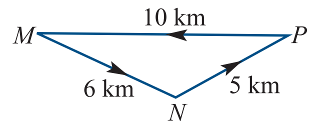
A $\sqrt{40}$ **B** $\sqrt{50}$ **C** $\sqrt{60}$
D $\sqrt{200}$ **E** $\sqrt{1000}$

38 In triangle ABC as shown, $\sin x = \frac{3}{7}$. The value of $\sin y$ is:



A $\frac{1}{7}$ **B** $\frac{9}{28}$ **C** $\frac{1}{2}$
D $\frac{4}{7}$ **E** $\frac{3}{4}$

39 A yacht follows a triangular course, *MNP*, as shown. The largest angle between any two legs of the course is closest to:

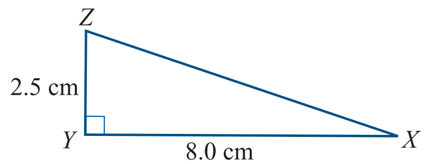


A 50° **B** 70° **C** 120°
D 130° **E** 140°

40 A hiker travels a distance of 5 km from point P to point Q on a bearing of 030°. She then travels from point Q to point R on a bearing of 330° for 10 km. How far *west* of P is R in kilometres?

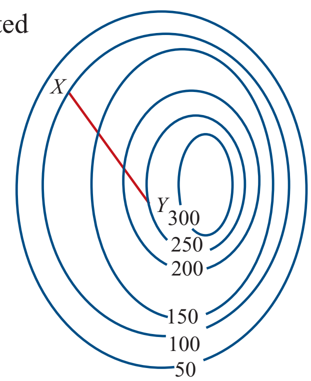
A 2.5 **B** 5 **C** 7.5 **D** 10 **E** 15

41 In right-angled triangle XYZ, $XY = 8.0$ cm and $YZ = 2.5$ cm as shown. The length of ZX, in centimetres, correct to one decimal place, is:



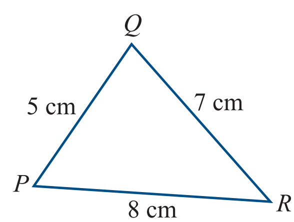
A 5.5 **B** 7.6 **C** 8.2
D 8.4 **E** 10.5

42 The contour map has scale 1 : 20 000. A path, XY, is represented on the map by a straight line segment 4 cm long. Point X is on the 100 metre contour and point Y on the 250 metre contour. The average slope of the path XY is:



A 0.03 **B** 0.075 **C** 0.1875
D 0.3125 **E** 0.375

43 In the triangle shown, angle PQR, correct to the nearest degree, equals:

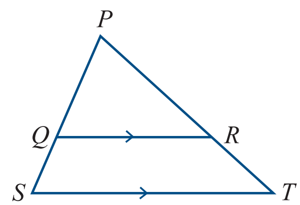


A 38° **B** 60° **C** 73°
D 82° **E** 98°

44 The diameter of a large sphere is 4 times the diameter of a smaller sphere. It follows that the ratio of the volume of the large sphere to the volume of the smaller sphere is:

A 4 : 1 **B** 8 : 1 **C** 16 : 1 **D** 32 : 1 **E** 64 : 1

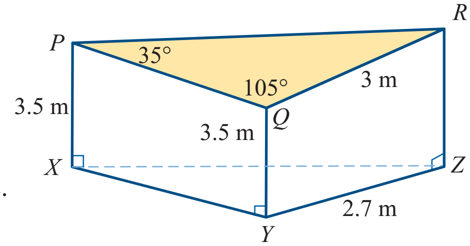
45 QR is parallel to ST and $PQ = 2QS$. Given that the area of triangle PST is 18 square centimetres, the area of triangle PQR in square centimetres is:



A 2 **B** 6 **C** 8
D 9 **E** 12

e Calculate the length PR. Write your answer in metres, correct to two decimal places.

The second piece of shade cloth PQR is attached to three vertical poles located at X, Y and Z as shown in the diagram. Poles PX and QY are each 3.5 metres long. The horizontal distance YZ is 2.7 metres.



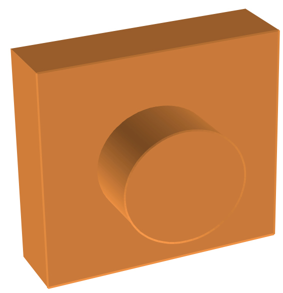
f Calculate the length of the vertical pole RZ. Write your answer correct to the nearest centimetre.

[VCAA pre 2006]

6 A wooden cylinder of radius 6 cm and height 5 cm is attached to a square flat wooden board of side length 12 cm and thickness 5 cm.

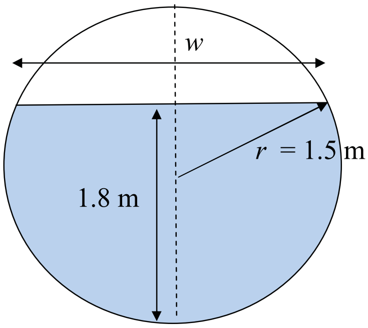
a Determine the volume of the composite shape correct to the nearest cm^3.

b Determine the total surface area of the composite shape correct to the nearest cm^2.



7 Two gliders travel in different directions from the same control tower.
Glider A travels 80 km on a bearing of 145°.
Glider B travels 50 km on a bearing of 055°.
Determine the bearing of glider A from glider B, correct to the nearest degree.

8 The cross-section of a waste water pipe is circular with a radius of 1.5 m, as shown. The water in the pipe is 1.8 m deep. Determine the width of the horizontal surface of the water (w). Give the answer in metres, correct to one decimal place.



1 Write down the equations and label them 1 and 2.

$$2x - y = 4 \quad (1)$$
$$x + 2y = -3 \quad (2)$$

2 To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

$$2x + 4y = -6 \quad (2')$$

3 Subtract (1) − (2′).

$$\therefore \quad -5y = 10 \quad (1) - (2')$$
$$\text{or } y = -2$$

4 Now substitute for y in (1) to find x.

$$\therefore \quad 2x - (-2) = 4$$

5 Check as in the substitution method.

$$\text{or } x = 1$$

How to solve simultaneous equations using the TI-Nspire CAS

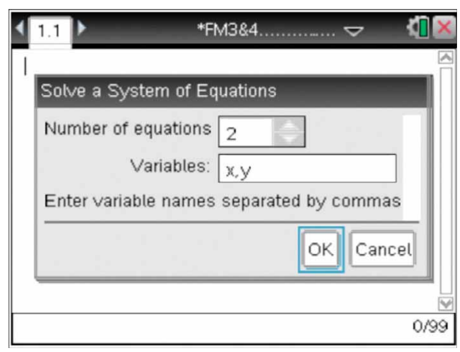
Solve the following pair of simultaneous equations:

$$24x + 12y = 36$$
$$45x + 30y = 90$$

Steps

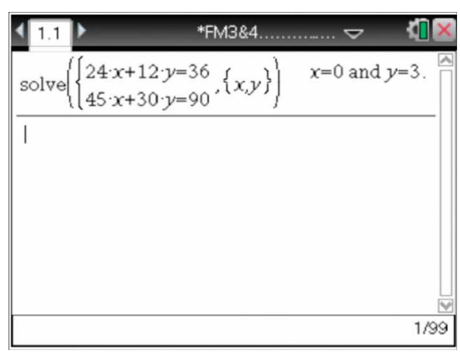
1 Start a new document and select **Add Calculator**.

2 Press (menu) **>Algebra>Solve System of Equations>Solve System of Equations** and complete the pop-up screen as shown (the default settings are for two equations with variables x & y).
A **simultaneous equation template** will be pasted to the screen.



3 Enter the equations into the template, as shown.

4 Press (enter) to display the solution, $x = 0$ and $y = 3$.



5 The solution $x = 0$ and $y = 3$ can be checked by substitution.

$$24 \times 0 + 12 \times 3 = 36 \checkmark$$
$$45 \times 0 + 30 \times 3 = 90 \checkmark$$

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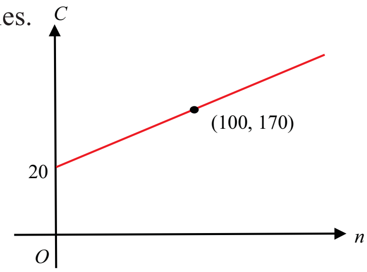
20 Given the straight line $2x + 3y - 6 = 0$, which of the following statements is *not* true?

A If x increases, then y decreases.
B The line has a negative gradient.
C The gradient of the line is -2.
D The line intercepts the x-axis at 3 and the y-axis at 2.
E The line is parallel to the line with the equation $2x + 3y - 4 = 0$.

21 The graph opposite shows the cost, $\$C$, of making n apple pies.
The profit from the sale of 80 apple pies is \$100.
The selling price of one apple pie is:

A \$1.50 **B** \$1.75 **C** \$2.50
D \$2.75 **E** \$3.75



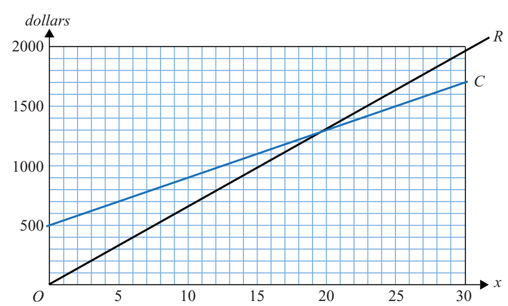
Extended-response question

1 Anne sells Softsleep pillows for \$65 each.

a Write an equation for the revenue, R dollars, that Anne receives from the sale of x Softsleep pillows.

b The cost, C dollars, of making Softsleep pillows is given by $C = 500 + 40x$. Find the cost of making 30 Softsleep pillows.

The revenue, R, from the sale of x Softsleep pillows is graphed opposite. Also shown is the graph of $C = 500 + 40x$.



c How many Softsleep pillows will Anne need to sell to break even?

[VCAA 2010]

Exercise 17D

1 Prepare a table of values and plot the graph of each of the following for $-3 \le x \le 3$. Use a graphics calculator to help you.

a $y = \frac{1}{2}x^2$ b $y = -\frac{1}{2}x^2$ c $y = \frac{4}{x^2}$ d $y = -\frac{4}{x^2}$ e $y = -\frac{2}{x}$

f $y = x^2$ g $y = -\frac{1}{2}x^3$ h $y = -3x$ i $y = -\frac{2}{3}x$

2 The point (1, 5) lies on the graph of $y = kx^3$. Find the value of k.

3 The point (2, 30) lies on the graph of $y = kx^2$. Find the value of k.

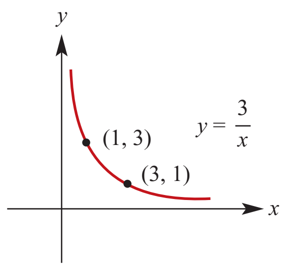
4 The point (16, 4) lies on the graph of $y = \dfrac{k}{x}$. Find the value of k.

5 The point (2, 1) lies on the graph of $y = \dfrac{k}{x^2}$. Find the value of k.

6 The point (3, −10) lies on the graph of $y = kx$. Find the value of k.

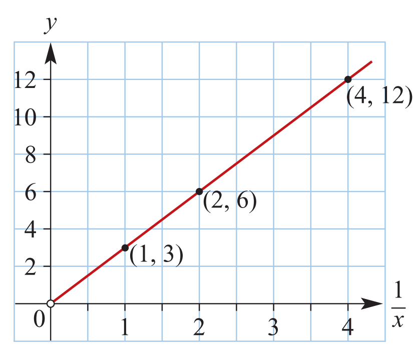
17.5 Linear representation of non-linear relations

Consider the relation with rule $y = \dfrac{3}{x}$ for $x > 0$.



A linear graph can be obtained by graphing y against $\dfrac{1}{x}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
$\frac{1}{x}$	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
y	12	6	3	$\frac{3}{2}$	1	$\frac{3}{4}$

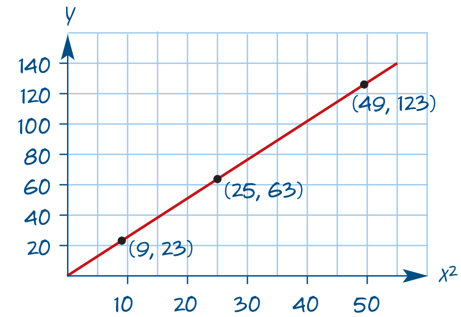


The gradient of the resulting straight line is 3.

Solution

a

x	0	1	3	5	7
x^2	0	1	9	25	49
y	0.5	3	23	63	123



b The y-axis intercept is 0.5.
The gradient of the straight line is 2.5.
The equation for the data is $y = 2.5x^2 + 0.5$.
Thus, the value of k is 2.5 and the value of c is 0.5.

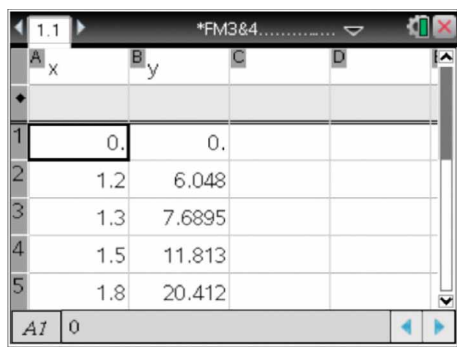
How to find a rule connecting x and y using the TI-Nspire CAS

Find a rule connecting x and y.

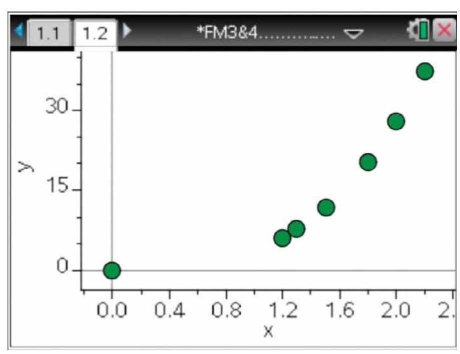
x	0	1.2	1.3	1.5	1.8	2	2.2
y	0	6.0480	7.6895	11.8130	20.4120	28.0000	37.268

Steps

1 Start a new document by pressing ctrl + N, and select **Add Lists & Spreadsheet**.
Enter the data into lists named x and y.



2 Press ctrl + I and select **Add Data & Statistics**. Construct a scatterplot of y against x. Let x be the independent variable and y the dependent variable.
The graph suggests that the rule is of the form $y = kx^n$.
To find the value of n we plot y against x^n. We are looking for a value of n that linearises the graph of y against x^n.
We will guess that the rule is $y = kx^3$. To test, we will plot y against x^3.



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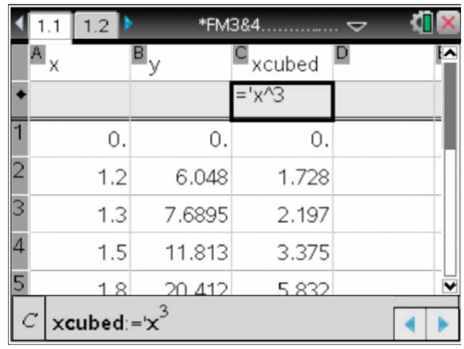
3 Return to **Lists & Spreadsheet** by pressing (ctrl + ◀).

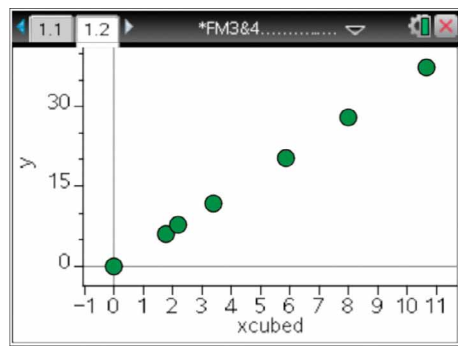


a Move the cursor to the top of column C and type ***xcubed*** (this list name will represent x-cubed). Press (enter).

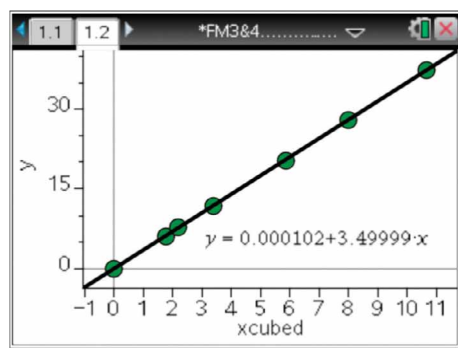
b Move the cursor to the grey cell immediately below the *xcubed* heading and type $=$. Then press **VAR** ((var)) and highlight the variable $\boldsymbol{x}$ and press (enter) to paste into the formula line. Press (^) (3), then press (enter) to calculate and display the cubed values.

Note: You can also type in the variable x and then select **Variable Reference** when prompted. This prompt occurs because $\boldsymbol{x}$ can also be a column name.

4 Construct a scatterplot of y against x^3. Use (ctrl) + ▶ to return to the scatterplot created earlier and change the independent variable to *xcubed*. The plot of y against x^3 is linear, suggesting that our guess that the rule is of the form $y = kx^3$ is correct.



5 To find the value of n we need to fit a line to our plot and find its slope, y against x^3. We do this by fitting a least squares (regression) line to the data. Press (menu) **>Analyze>Regression>Show Linear (ax+b)**. The equation is $y = 3.5x^3$, correct to 1 decimal place.



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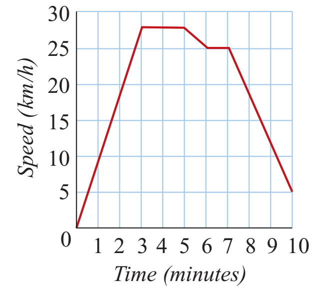
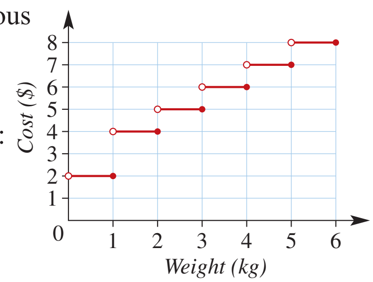
12 The graph shows the speed of a cyclist over ten minutes of a journey. The cyclist's speed decreased most rapidly in the period between:



A 0 and 3 minutes **B** 3 and 5 minutes
C 5 and 6 minutes **D** 6 and 7 minutes
E 7 and 10 minutes

13 The graph shows the cost of posting parcels of various weights. A person posts two parcels, one weighing 3 kg and the other 1.5 kg. If each parcel is charged for separately, the cost of sending the two parcels is:



A \$4.50 **B** \$5.00 **C** \$7.00
D \$9.00 **E** \$10.00

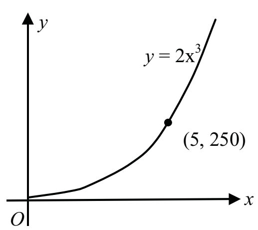
14 The cost (\$$C$) of renting a car is given by $C = an + b$ where:

- n is the number of kilometers travelled
- a is the cost per kilometres travelled and
- b is a fixed cost.

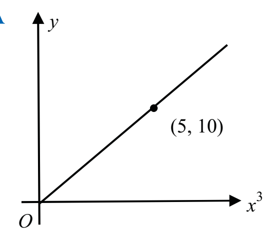
For a person travelling 200 km, the cost of car rental is \$430. For a person travelling 315 km, the cost of car rental is \$660. The values of a and b are:

A $a = 1, b = 230$ **B** $a = 2, b = 30$ **C** $a = 3, b = 10$
D $a = 200, b = 430$ **E** $a = 315, b = 660$

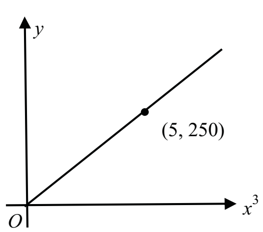
15 The graph of $y = 2x^3$ is shown below. Which of the graphs A to C can also be used to represent this relationship between x and y?



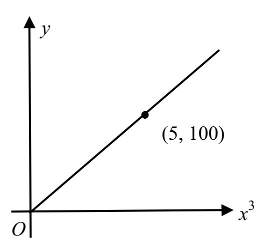
A



B



C



Extended-response questions

1 The Winkle Winery makes white wine that is sold at the cellar door for \$15 per bottle. The winery has fixed costs that total \$3200 per week. In addition, it costs \$7 to produce each bottle of wine.

a Find the total cost (including the fixed costs) of producing 200 bottles of white wine in a week.

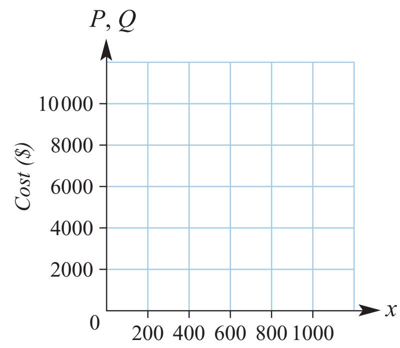
b Find the revenue obtained by the winery for the sale of 200 bottles of white wine.

c Draw up a grid like the one shown, and draw the graphs of the rules for:

i \$$P$, the total cost of producing x bottles of white wine in a week

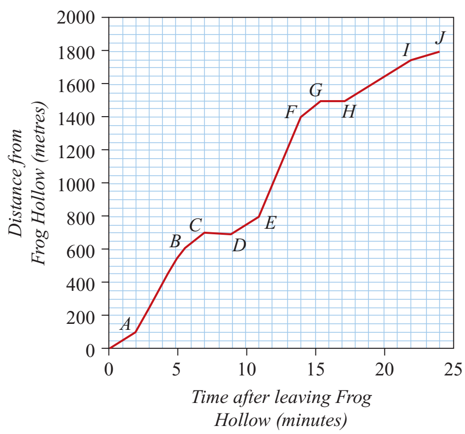
ii \$$Q$, the total amount of money received by selling x bottles of white wine in a week

d Using the graph, or otherwise, find the smallest number of bottles of wine that the Winkle Winery needs to sell in a week in order to break even (to cover all costs).



[VCAA pre 2006]

2 At the Gum Flat Fun Park there are many attractions. One that appeals especially to the younger visitors is the train Puffing Polly. The distance–time graph represents a train trip for Puffing Polly from Frog Hollow to Eagle Hill, stopping at two stations on the way.



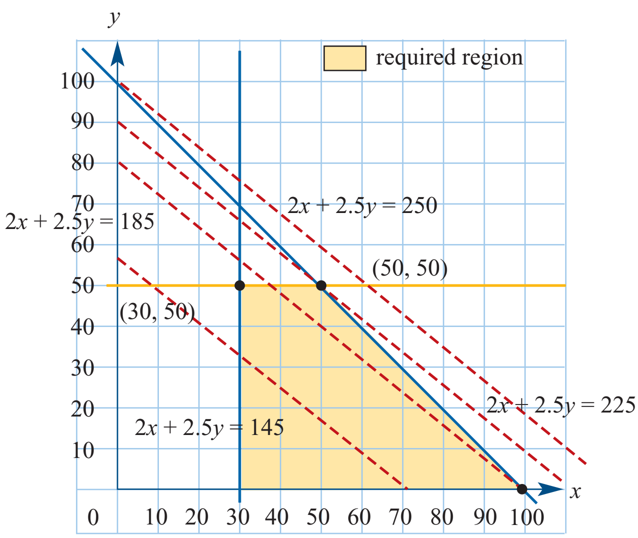
a What is the total time for which Puffing Polly is stopped at the two stations?

b **i** Which line segment of the graph represents the section of the trip when Puffing Polly is travelling fastest?

ii Find Puffing Polly's speed for this section of the trip, clearly stating the units used in your answer.

[VCAA pre 2006]

There are other values of x and y that yield a profit of \$145
e.g. $x = 55$, $y = 14$; $x = 50$, $y = 18$; $x = 45$, $y = 22$; $x = 30$, $y = 34$.
All these points lie on the straight line $2x + 2.5y = 145$ (a profit of \$145).
For a profit of \$185, the straight line is $2x + 2.5y = 185$.
For a profit of \$225, the straight line is $2x + 2.5y = 225$.
All these lines are parallel. They have been added to the graph of the feasible region.



- It can be seen that the maximum profit possible is \$225 and this is achievable at only one point in the feasible region; i.e. (50, 50).
- It can be seen that a larger value of P (e.g. $P = 250$) will not yield points in the feasible region.

The function with the rule $P = 2x + 2.5y$ is an example of an **objective function**. In linear programming problems, the aim is to find the maximum or minimum value of an objective function for a given feasible region. To help us do this, we can make use of the **corner point principle**.

The corner point principle
In linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region.
Note: If two corners give the same maximum or minimum value, then all points along a line joining the two points will also have the same maximum or minimum values. This occurs when the family of lines produced by the objective function is parallel to one of the boundaries of the feasible region.

This means that we need to evaluate the objective function only at each of the corner points to find the maximum or minimum value of an objective function.

b The crosses in the feasible region indicate the possible numbers of small and large vans.

c **i** The largest number of vehicles that could be used is 9: either 5 large and 4 small or 4 large and 5 small or 6 large and 3 small or 7 large and 2 small.

ii The smallest number of vehicles that could be used is 6 large vans.

Example 9 Setting up and solving a linear programming problem

A business produces imitation antique vases in two styles: Ming Dynasty vases and Geometric Period Greek vases.

Each vase requires:

- potters to make the vase
- artists to decorate the vase.

During one week the business employs 10 potters and 4 artists. Each employee works for a total of 40 hours from Monday to Friday.

The time spent making each vase is shown in the table below.

Employee	*Ming*	*Geometric*
Potter	8 hours	4 hours
Artist	2 hours	2 hours

Let x represent the number of Ming Dynasty vases made in a week and let y represent the number of Geometric Period vases made in a week.

To meet regular orders, the business must make at least 10 Ming vases and 20 Geometric vases each week. These constraints can be written as:

$$x \geq 10 \text{ and } y \geq 20$$

The total time available to the *potters* to make vases is 400 hours. This constraint can be written as:

$$8x + 4y \leq 400$$

a The total time available to the *artists* is 160 hours. Write an equality to represent this constraint.

b Draw the graphs of the four inequalities in part **a**.

c Shade in the feasible region on the graph.

The profit is \$50 on each Ming vase and \$30 on each Geometric vase.

d All vases produced in a week are sold. Write down an expression in terms of x and y for the total profit, \$$P$, that the business will receive.

e Determine the number of each type of vase that should be produced in a week to result in the maximum profit.

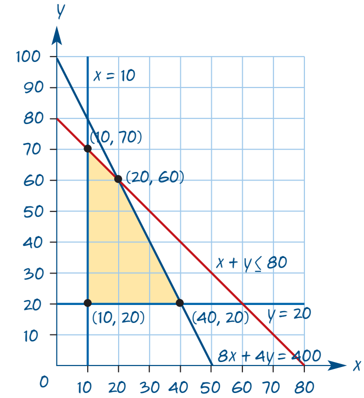
Due to increased costs, the profit made on each Geometric vase is decreased from \$30 to \$25. As a result, the profit from making x Ming vases and y Geometric vases is now given by $P = 50x + 25y$.

f Determine the number of each type of vase that should now be produced to result in the maximum profit.

Solution

a $2x + 2y \le 160$ or, equivalently, $x + y \le 80$

b, c



d $P = 50x + 30y$

e Evaluate at each of the vertices of the feasible region:

$$(10, 20)\ P = 50 \times 10 + 30 \times 20 = 1100$$
$$(10, 70)\ P = 50 \times 10 + 30 \times 70 = 2600$$
$$(20, 60)\ P = 50 \times 20 + 30 \times 60 = 2800$$
$$(40, 20)\ P = 50 \times 40 + 30 \times 20 = 2600$$

To maximise the profit, 20 Ming vases and 60 Geometric vases should be produced.

f $P = 50x + 25y$

Evaluate P at each of the vertices of the feasible region:

$$(10, 20)\ P = 50 \times 10 + 25 \times 20 = 1000$$
$$(10, 70)\ P = 50 \times 10 + 25 \times 70 = 2250$$
$$(20, 60)\ P = 50 \times 20 + 25 \times 60 = 2500$$
$$(40, 20)\ P = 50 \times 40 + 25 \times 20 = 2500$$

The maximum value of $P = 2500$ occurs at two vertices, (20, 60) and (40, 20). However, the corner point principle tells us that any point on the line joining these two vertices is also a solution. But, because we can only accept integer solutions, we can see from the graph that there is only one other integer solution and that occurs at the point (30, 40). Evaluating P at this point confirms that the value of P is also maximised at this point:
$P = 50 \times 30 + 25 \times 40 = 2500$.
Thus the profit can be maximised in three ways: by producing 20 Ming vases and 60 Geometric vases , as before; by producing 40 Ming vases and 20 Geometric vases; or by producing 30 Ming vases and 20 Geometric vases.

Exercise 18D

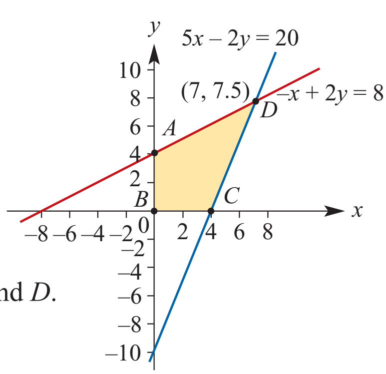
Skillsheet TEACHER

Constructing feasible regions and optimising objective functions

1 The region that satisfies all of the following constraints:

$$5x - 2y \le 20$$
$$-x + 2y \le 8$$
$$x \ge 0$$
$$y \ge 0$$

is as shown.

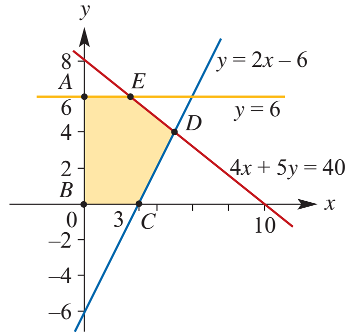


a Write down the values of the coordinates of A, B, C and D.

b Find the maximum value of $z = x + 2y$ subject to the set of constraints above.

2 The region that satisfies all the following constraints is shown:

$$4x + 5y \le 40$$
$$y \ge 2x - 6$$
$$x \ge 0$$
$$0 \le y \le 6$$



a Find the coordinates of A, B, C, D and E.

b Find the maximum value of $z = 2x + y$ subject to the set of constraints above.

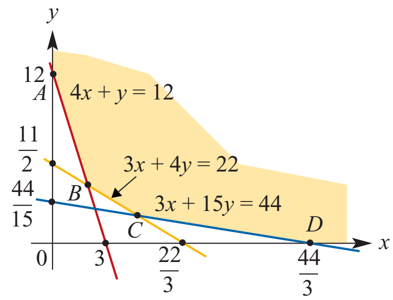
3 a Illustrate the region that satisfies all the following constraints:

$$x + 3y \le 17$$
$$5x + 3y \ge 25$$
$$0 \le x \le 8$$
$$0 \le y \le 6$$

b Find the maximum value of $z = x + 3y$ subject to the set of constraints in **a**.

4 The region that satisfies the following constraints is shown:

$$4x + y \ge 12$$
$$3x + 4y \ge 22$$
$$3x + 15y \ge 44$$
$$x \ge 0$$
$$y \ge 0$$



a Find the coordinates of points A, B, C and D.

b Find the minimum value of $z = 3x + 2y$ subject to the set of constraints above.

5 a Illustrate the region that satisfies all the following constraints:

$$4x + 5y \ge 52$$
$$y \ge 0.5x$$
$$y \le 1.8x$$
$$x \ge 4$$
$$y \ge 0$$

b Find the minimum value of $z = 4x + 10y$ subject to the set of constraints in **a**.

Setting up and solving linear programming problems (constraints given)

6 Samantha wants to buy some CDs that cost \$13 each and some books that cost \$12 each. She wants to buy more than two CDs and more than five books, but can spend no more than \$156.

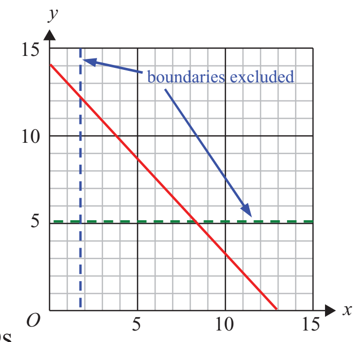
Let x be the number of CDs and y be the number of books that Samantha buys.

The three constraints than x and y must satisfy are:

Constraint 1: $13x + 12y \le 156$

Constraint 2: $x > 2$

Constraint 3: $y > 5$



a Explain the meaning of Constraint 1 in the context of this problem. The lines defining these constraints have been plotted on the graph shown.

b Identify the feasible region by shading.

c Write down the coordinates of the six points in the feasible region that satisfy these constraints.

d Why is buying two CDs and eight books not a possible solution?

e Samantha wants to maximise T, the total number of CDs and books she buys. Write down an expression for T in terms of x and y.

f Noting that x and y can only take integer values, determine the maximum total number of CDs and books Samantha can buy. List all possible combinations of CDs and books she can buy to give this maximum.

7 An ice cream manufacturer makes just two flavours: chocolate superb and vanilla royal. Past experience shows that she needs to make at least 1200 litres of chocolate ice cream and at least 600 litres of vanilla ice cream each day. The maximum amount of ice cream she make can each day (chocolate + vanilla) is 2000 litres.
Let x be the amount (in litres) of chocolate ice cream she makes each day and y be the amount (in litres) of vanilla ice cream she makes each day.
The three constraints that x and y must satisfy are:

Constraint 1: $x \geq 1200$
Constraint 2: $y \geq 600$
Constraint 3: $x + y \leq 2000$

a Explain the meaning of Constraint 3 in the context of this problem.

b Graph these three constraints and identify the feasible region. Determine the coordinates of each of the corner points of the feasible region.

The profit from the sale of one litre of chocolate ice cream is \$1.10 and the profit from the sale of one litre of vanilla ice cream is \$0.95.

c Write down an expression for the profit, \$$P$, the ice cream manufacturer will make by selling x litres of chocolate ice cream and y litres of vanilla ice cream.

d Determine the amounts of chocolate and vanilla ice cream she should make each day to maximise her profit. Determine this profit.

The icecream manufacturer changes her prices so that the profit she makes on both types of ice cream is \$1.00.

e Write down an expression for the profit, \$$P$, the ice cream manufacturer will now make by selling x litres of chocolate ice cream and y litres of vanilla ice cream.

f Determine the amounts of chocolate and vanilla ice cream she should now make each day to maximise her profit and determine this profit.

8 A mining company is required to move 200 workers and 36 tonnes of equipment by air. It is able to charter two aircraft: a Hawk, which can accommodate 20 workers and 6 tonnes of equipment; and an Eagle, which can accommodate 40 workers and 4 tonnes of equipment.

Let x denote the number of trips made by the Hawk aircraft and let y denote the number of trips made by the Eagle aircraft. The four constraints on the values x and y can take are:

Constraints 1 and 2: $x \geq 0$ and $y \geq 0$
Constraint 3: $20x + 40y \geq 200$
Constraint 4: $6x + 4y \geq 36$

a Graph these constraints and identify the feasible region.

Hawk aircraft cost \$3000 per trip while Eagle aircraft cost \$4000 per trip.

b Write down an expression for the cost, \$$C$, of making x trips with a Hawk aircraft and y trips with an Eagle aircraft.

c Determine the number of trips that should be made by each of the aircraft to mimimise the total cost. Determine this cost.

9 For a journey across Ellesmene Island in the Arctic Circle, an explorer wishes to travel as lightly as possible. He can obtain supplies of two types of lightweight food, type X and type Y, with energy, protein and carbohydrate contents shown in the table.

Food	*Energy per serve (units)*	*Protein per serve (units)*	*Carbohydrate per serve (units)*
Type X	600	3.0	20
Type Y	400	3.5	35

The explorer estimates that his minimum daily requirements of energy, protein and carbohydrate will be 2600 units, 19 units and 150 units, respectively. Each serve of type X food weighs 36 g and each serve of type Y food weighs 56 g.

If x and y are the number of serves per day of type X and type Y foods, respectively, that the explorer takes, four of the constraints on x and y are:

$$x \geq 0 \qquad y \geq 0$$
$$600x + 400y \geq 2600$$
$$3x + 3.5y \geq 19$$

a Give the constraint determined by the amount of carbohydrate required.

b If W grams is the weight per day of food that the explorer takes, write down the rule for W in terms of x and y.

c Determine the number of serves per day of each food type the explorer should take to minimise weight while still satisfying his daily dietary requirements.

Setting up constraints and objective functions

10 A market gardener decides to buy fertiliser to spread on her garden beds. Two types of fertiliser are available: Fast Grow and Easy Grow. Each fertiliser contains three different types of nutrients (A, B, and C) that promote plant growth but in different amounts.

Nutrient	*Fast Grow*	*Easy Grow*	*Minimum nutrient requirement*
A	3 units/bag	2 units/bag	160 units
B	5 units/bag	2 units/bag	200 units
C	1 unit/bag	2 units/bag	80 units

Let x be the number of bags of Fast Grow and y be the number of bags of Easy Grow.

a There must be at least 160 units of nutrient A. Write down an inequality in terms of x and y that can be used to represent this constraint.

b There must be at least 200 units of nutrient B. Write down an inequality in terms of x and y that can be used to represent this constraint.

c There must be at least 80 units of nutrient C. Write down an inequality in terms of x and y that can be used to represent this constraint.

d The cost of a bag of Fast Grow is \$4.00 while the cost of a bag of Easy Grow is \$3.00. Write down an expression for the cost, \$$C$, of buying x bags of Fast Grow and y bags of Easy Grow.

11 A small service station sells petrol and diesel. The owner needs to refill his tanks. There are a number of constraints that determine how much petrol and diesel he orders. Let x be the volume (in litres) of diesel and y be the volume (in litres) of petrol he orders.
Constraints 1 & 2: The maximum amount of diesel he can store in his tanks is 15 000 litres. The maximum amount of petrol he can store in his tanks is 20 000 litres.

a Write down two inequalities in terms of x and y that can be used to represent these constraints.

Constraint 3: Experience shows that the demand for petrol is at least double the demand for diesel. That is, for every litre of diesel he has in his tanks, he needs to have at least two litres of petrol.

b Write down an inequality in terms of x and y that represents this constraint.

Constraint 4: One litre of diesel costs the service station \$1.20 while one litre of petrol costs the service station \$1.12. The service station can afford a maximum of \$25 000 in total to pay for the fuel.

c Write down an inequality in terms of x and y that represents this constraint.

Objective function: The profit made from the sale of one litre of diesel is 5.5 cents while the profit made from the sale of one litre of petrol is 4.6 cents.

d Write down an expression for the profit, \$$P$, the service station will make by selling x litres of diesel and y litres of petrol.

12 A factory makes two products, Widgets and Gidgets, each of which is constructed using different numbers of three different component parts, A, B and C.

Component part	*Widgets*	*Gidgets*	*Number in stock*
A	12	9	280
B	8	11	260
C	10	13	320

The number of Widgets and Gidgets that can be made is constrained by the number of As, Bs and Cs currently in stock. These numbers are also shown in the table.
Let x be the number of Widgets made each month.
Let y be the number of Gidgets made each month.
One of the three constraints that x and y must satisfy is:
Constraint 1: $12x + 9y \le 280$

a Explain the meaning of Constraint 1 in the context of this problem.

Constraint 2 arises because the total number of *B*s used cannot exceed 260.

Constraint 3 arises because the total number of *C*s used cannot exceed 320.

b Write down two inequalities in terms of x and y that can be used to represent constraints 2 and 3.

The profit from the sale of one Widget is \$84 while the profit from the sale of one Gidget is \$72.

c Write down an expression for the profit, \$$P$, the factory will make from producing x Widgets and y Gidgets.

By using new technology, the factory is able to increase the profit it makes from the sale of one Widget to \$120 and the profit it makes from the sale of one Gidget to \$90.

d Write down an expression for the profit, \$$P$, the factory will make from producing x Widgets and y Gidgets under these new circumstances.

Setting up and solving linear programming problems

13 An outdoor clothing manufacturer has 520 metres of polarfleece material. The manufacturer will use it to make jackets of two types, Polarbear and Polarfox, to sell to retailers. For each jacket of either type, 2.0 metres of material is required. However, the Polarbear is simpler in design, requiring 2.4 hours each in the production process, while each Polarfox requires 3.2 hours. There are 672 hours available.

From past experience of demand, the manufacturer has decided to make no more than half as many Polarfox jackets as Polarbear jackets. If the profit on each Polarbear jacket is \$36 and the profit on each Polarfox jacket is \$42, use a graphical method to find how many of each type should be made in order to maximise profit. What is this maximum profit?

14 The army is required to airlift 450 people and 36 tonnes of baggage by helicopter. There are 9 Redhawk helicopters and 6 Blackjet helicopters available. Each Redhawk can carry 45 passengers and 3 tonnes of baggage, while each Blackjet can carry 30 passengers and 4 tonnes of baggage. Running costs per hour are \$1800 for each Redhawk and \$1600 for each Blackjet.

If the army wishes to minimise the cost of the airlift per hour, use a graphical method to find how many of each helicopter should be used.

MC TEST D&D TEST

Review

c Use the graph from Question 1 and the expression from Question 2b, to determine the number of models of each type that should be produced in the week to result in the largest possible profit for Adam.

d Find the maximum profit that Adam could make in the week. [VCAA pre 2006]

3 A small canning company produces two types of canned tuna with additional chilli: Super Tuna with fried chilli and Elite Tuna with dried chilli. A can of Super Tuna requires 200 g of tuna and 30 g of chilli. A can of Elite Tuna requires 300 g of tuna and 20 g of chilli. The company can produce 800 cans of tuna a day. There is 200 kg of tuna available to them every day and no limit on the chilli. Let x be the number of cans of Elite Tuna produced in a day and let y be the number of cans of Super Tuna produced in a day. The inequalities that represent these constraints are:

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 800$$
$$0.3y + 0.2x \leq 200$$

a Sketch the graphs of these inequalities.

b Find the coordinates of the vertices of the region defined by these inequalities.

c If the company makes \$1.00 profit on the Elite and \$0.80 profit on the Super, write an expression for the daily profit in terms of x and y.

d How many cans of each type of tuna should be produced to maximise the profit?

4 The Victory Vineyard makes both a red wine and a white wine. The table summarises the production costs and sale prices per bottle of the white wine and the red wine.

Type of wine	*Production cost*	*Sale price*
White	\$7	\$15
Red	\$10	\$20

The following constraints apply to the production of wine at the Victory Vineyard.

- The maximum total number of bottles of red wine and white wine produced by the Victory Vineyard in any day is 700.
- The total production cost of red and white wine cannot exceed \$6400 per day.
- The maximum number of bottles of red wine that can be produced is 570 per day.

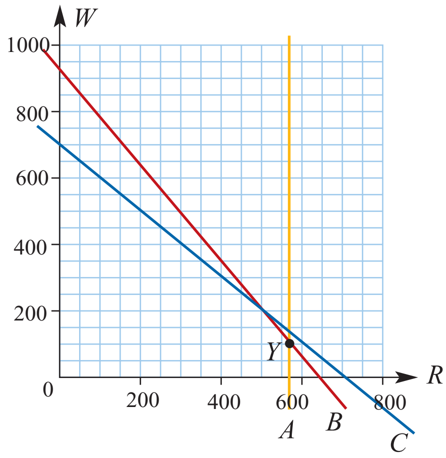
Use R to represent the number of bottles of red wine produced each day and W to represent the number of bottles of white wine produced each day. These conditions are then expressed algebraically by:

Constraint 1 $R + W \leq 700$

Constraint 2 $10R + 7W \leq 6400$

Constraint 3 $R \leq 570$

a The lines defined by these constraints are shown on the graph that follows. Identify the lines on this graph that define the boundaries of each of these constraints.



b Find the co-ordinates of the point Y on the graph.

c The profit per bottle is \$8 for white wine and \$10 for red wine. Write down an expression for the profit P in terms of W and R.

d Find the maximum daily profit that can be earned by the Victory Vineyard from selling their wine. [VCAA pre 2006]

5 Harry offers dog washing and dog clipping services.
Let x be the number of dogs washed and y be the number of dogs clipped in one day.
It takes 20 minutes to wash a dog and 25 minutes to clip a dog. There are 200 minutes available each day to wash and clip dogs.
This information is represented by Inequalities 1–3.

Inequality 1: $x \geq 0$ Inequality 2: $y \geq 0$ Inequality 3: $20x + 25y \leq 200$

In any one day the number of dogs clipped is *at least* twice the number of dogs washed.

a Write an inequality (Inequality 4) to describe this information in terms of x and y.

b **i** Using graph paper, draw in the boundaries of the feasible region defined by Inequalities 1–4.

ii On a day when exactly five dogs are clipped, what is the maximum number of dogs that could be washed?

The profit from washing one dog is \$40 and the profit from clipping one dog is \$30.
Let P be the total profit obtained in one day from washing and clipping dogs.

c Write an equation for the total profit, P, in terms of x and y.

d **i** Determine the number of dogs that should be washed and the number of dogs that should be clipped in one day in order to maximise the total profit.

ii What is the maximum total profit generated by washing and clipping dogs in one day?

[VCAA 2006]

CHAPTER

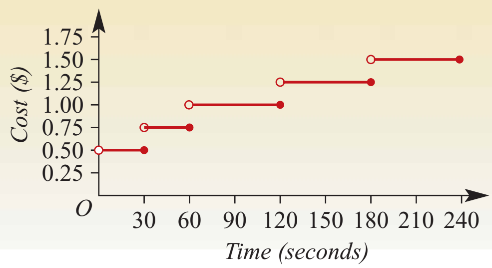
19

MODULE 3

Revision: Graphs and relations

19.1 Multiple-choice questions

1 The graph shows the cost (dollars) of mobile telephone calls up to 240 seconds long. The cost of making a 90-second call followed by a 30-second call is:



A \$1.00 **B** \$1.20 **C** \$1.25
D \$1.50 **E** \$1.75

2 The point (2, 1) lies on the line $y = 3x + c$. The value of c is:

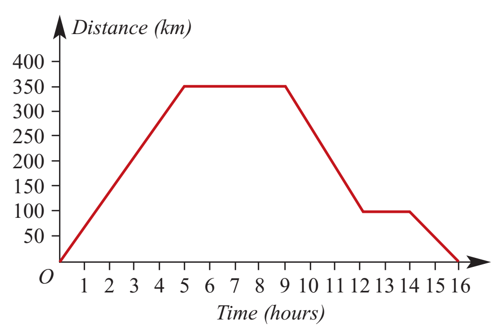
A -7 **B** -5 **C** -1 **D** 5 **E** 7

3 The lines $y + 8 = 0$ and $x - 12 = 0$ intersect at the point:

A $(-12, 8)$ **B** $(-8, 12)$ **C** $(0, 0)$ **D** $(8, -12)$ **E** $(12, -8)$

Questions 4 and 5 refer to the following graph.

The graph shows a distance–time graph for a car travelling from home along a long straight road over a 16-hour period.



4 In which one of the time intervals is the speed of the car greatest?

A 0 to 5 hours **B** 5 to 9 hours
C 9 to 12 hours **D** 12 to 14 hours
E 14 to 16 hours

5 After 12 hours the car has travelled a total distance of:

A 100 km **B** 350 km **C** 450 km **D** 600 km **E** 700 km

e Materials for each Elite cost \$1200 and for each Sprint cost \$700. The cost of labour is \$20 per person-hour. Taking only these costs into account, write down the cost of making each:

i Elite bicycle **ii** Sprint bicycle

f In addition to the costs of materials and labour, it costs \$10 000 per month to run the triathlon bicycle manufacturing business. Emma and Brad decide to make only 6 Sprint bicycles in the coming month. Taking all costs into account, write down:

i the cost, \$$C$, of making the 6 Sprint bicycles and x Elite bicycles in a month

ii the total revenue, \$$T$, made from selling the 6 Sprint bicycles and x Elite bicycles produced during the month.

g If they sell the six Sprint bicycles produced during the month, what is the minimum number of Elite bicycles they would need to produce and sell to avoid making a loss for the month? Comment on this number in terms of the given constraints. [VCAA pre 2006]

5 A manufacturer makes furniture at two factories, one in Melbourne and the other in Wangaratta.

Let x be the number of hours per week during which the factory in Melbourne makes lounge chairs.

Let y be the number of hours per week during which the factory in Wangaratta makes lounge chairs.

In Melbourne, 40 hours per week are available for making lounge chairs.

In Wangaratta, 35 hours per week are available for making lounge chairs.

a Write down two inequalities that express these constraints on x and y.

Three types of lounge chair are made: the Deluxe, the Standard and the Convertible.

At the Melbourne factory, 20 Standard, 8 Deluxe and 4 Convertible lounge chairs can be made per week.

At the Wangaratta factory, 20 Standard, 4 Deluxe and 24 Convertible lounge chairs can be made per week.

Type	*Number produced in a week*
Standard	$20x + 20y$
Deluxe	$8x + 4y$
Convertible	$4x + 24y$

In terms of x and y, the number of each type of lounge chair that can be assembled in a week is given in the table.

To meet demand, the furniture manufacturer must produce at least 400 Standard, 120 Deluxe and 160 Convertible lounge chairs each week.

b Write down the constraints that arise from these requirements.

c On a set of axes, draw a graph for the complete set of constraints of parts **a** and **b**, clearly indicating the feasible region. Use a scale of 0 to 40 for x and y.

d The operating costs in Melbourne are \$80 per hour and in Wangaratta \$60 per hour. If \$$C$ is the total operating cost per week of the two factories, express C in terms of x and y.

e Find the number of hours per week for which each factory should operate to make lounge chairs so that the manufacturer can satisfy the demand for the chairs at the lowest possible operating cost.

Since the amount of simple interest earned is the same every year, we can apply a general rule.

Simple interest

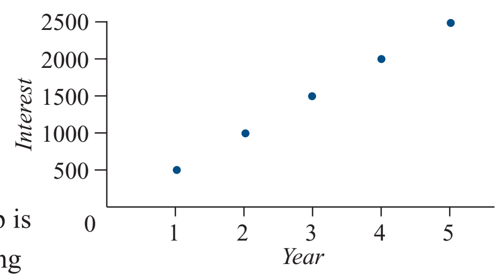
To calculate the simple interest earned or paid use the rule

$$I = \frac{Prt}{100}$$

where

P = amount invested or borrowed
r = interest rate per annum
t = is the time in years

How does this relationship look graphically? Suppose that we were to borrow \$5000 at 10% per annum simple interest for a period of years. A plot of **interest** against **time** is shown.



From this graph we can see that the relationship is linear, with the amount of interest paid or due being directly proportional to the time for which the money is borrowed or invested. The slope or gradient of a line that could be drawn through these points is numerically equal to the interest rate.

To determine the amount of the investment, the interest earned is added to the amount initially invested.

The amount of an investment, A, is the principal plus the amount of interest earned.

$$A = P + I = P + \frac{Prt}{100}$$

Here, P is the amount invested or borrowed, r is the interest rate and t is the time (in years).

If the money is invested for more or less than one year, the amount of interest payable is proportional to the length of time for which it is invested.

Example 6 **Calculating the simple interest for a period other than one year**

How much interest would be due on a loan of \$5000 at 10% per annum for six months?

Solution

Apply the formula with $P = 5000$, $r = 10\%$ and $t = 6/12$ since the investment is only for 6 months and the interest rate is for the whole year.

$$I = \frac{Prt}{100} = 5000 \times \frac{10}{100} \times \frac{6}{12}$$
$$= \$250$$

Example 7 **Calculating the total amount owed**

Find the total amount owed on a loan of \$10 000 at 12% per annum simple interest at the end of two years.

Solution

1 Apply the formula with $P = 10\,000$, $r = 12\%$ and $t = 2$ to find the interest.

$$I = \frac{Prt}{100} = 10\,000 \times \frac{12}{100} \times 2$$
$$= \$2400$$

2 Find the total owed by adding the interest to the principal.

$$A = P + I = 10\,000 + 2400 = \$12\,400$$

The formula given for simple interest can be rearranged to find any of the variables when the values of the other three variables are known.

Finding the interest rate

To find the interest rate per annum, r, given the values of P, I and t:

$$r = \frac{100I}{Pt}$$

where P is the principal, I is the amount of interest and t is the time in years.

Example 8 **Calculating the interest rate of the loan or investment**

Find the rate of simple interest charged per annum if a loan of \$20 000 incurs interest of \$12 000 after eight years.

Solution

1 Apply the formula with $P = 20\,000$, $I = 12\,000$ and $t = 8$ to find the value of r.

$$r = \frac{100I}{Pt} = \frac{100 \times 12\,000}{20\,000 \times 8}$$
$$= 7.5\%$$

2 Since the unit of time was years, the interest rate can be written as the interest per annum.

Interest rate $= 7.5\%$ per annum

Finding the term

To find the number of years or term of an investment, t years, given the values of P, I and r:

$$t = \frac{100I}{Pr}$$

where P is the principal, I is the amount of interest and r is the interest rate per annum.

How to use the graphics TI-Nspire CAS to solve simple interest problems

Suppose that we wish to know the length of time it would take for \$40 000 invested at 6.25% interest per annum to earn \$10 000 interest.

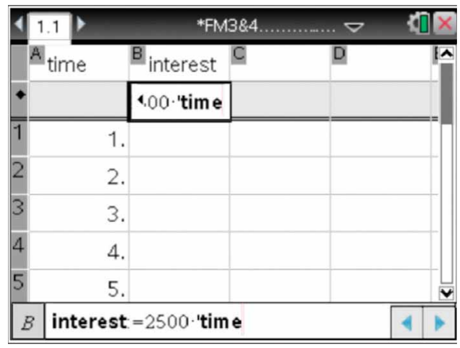
Steps

1 Substitute $P = 40\,000$, $r = 6.25$ into the formula for interest rate.

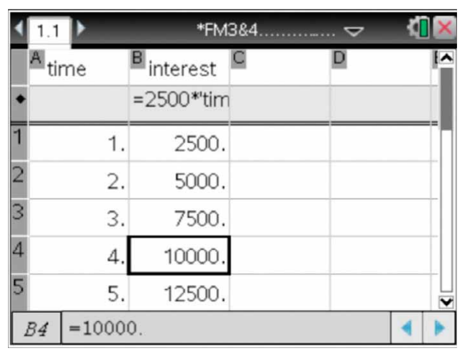
$$I = \frac{Prt}{100} = \frac{40\,000 \times 6.25 \times t}{100}$$
$$= 2500t$$

Method 1: Form a table

2 Start a new document by pressing (ctrl) + [N] and select **Add Lists & Spreadsheet**.
Name the lists *time* (to represent time in years) and *interest*.
Enter the data **1** to **10** into the list named *time*, as shown.
Note: You can also use the sequence command to do this.

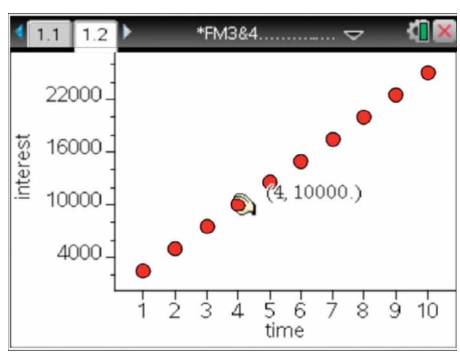


3 Place the cursor in the grey formula cell in the list named *interest* and type $=$ **2500** $\times$ **time**.
Note: You can also use the (var) key and paste *time* from the variable list.
Press (enter) to display the values as shown.
Scrolling down the table we can see that it takes 4 years to earn \$10 000 interest.



Method 2: Draw a graph

4 Press (ctrl) + [I] and select **Add Data & Statistics** and construct a scatterplot of *interest* against *time*, as shown.



Notes:

1 To connect the data points, press (ctrl) + (menu) and select **Connect Data Points**.

2 To display a value, place the cursor over the data point or use (menu) **>Analyze>Graph Trace**.

We can see that it takes 4 years to earn \$10 000 interest. It is also worth noting that the slope of the line is equal to the interest earned per year.

Note: You can also graph this example in the **Graphs** application and use the **Function Table** to answer key questions.

ISBN 978-1-107-65590-4

Cambridge University Press

Exercise 20B

Calculating the amount of interest earned or paid

1 Calculate the simple interest on the following amounts:

- **a** \$2000 invested at 6% per annum for four years
- **b** \$10 000 invested at 12% per annum for five years
- **c** \$8000 invested at 12.5% per annum for three years

2 Calculate the simple interest on the following amounts:

- **a** \$10 000 invested at 6% per annum for eight months
- **b** \$3500 invested at 10% per annum for 54 months
- **c** \$12 000 invested at 12% per annum for $1\frac{1}{4}$ years

3 Find the simple interest that is earned on the following investments:

- **a** \$1000 invested for one year at 6% per annum
- **b** \$5400 invested for three years at 7% per annum
- **c** \$875 invested for three-and-a-half years at 5% per annum

4 A sum of \$8500 was invested in a fixed-term deposit account for three years. Calculate the simple interest earned if the rate of interest is 7.9% per annum.

5 Find the amount of interest paid on a personal loan of \$7000 taken out at a simple interest rate of 14% per annum over a period of:

a 18 months **b** two years **c** three years and 150 days

6 Ben decides to invest his savings of \$1850 from his holiday job for five years at 13.25% per annum simple interest.

- **a** How much will he have at the end of this period?
- **b** Use your graphics calculator to sketch a graph of the simple interest earned against time (in years).

7 A loan of \$900 is taken out at a simple interest rate of 16.5% per annum.

- **a** How much is owing after four months have passed?
- **b** Use your graphics calculator to sketch a graph of the simple interest paid against time (in years).

8 For each of the following, calculate the interest payable and the balance of the account after one year:

- **a** \$500 deposited in a savings account at 3.5% per annum
- **b** \$1200 invested in a fixed term deposit account at 5.1% per annum
- **c** \$4350 transferred to an advantage saver account at 7% per annum

Calculating the simple interest rate

9 Find the rate of simple interest per annum used in the following investments:

a \$5300 invested for five years and earning \$2119 interest

b \$620 invested for one year and earning \$24 interest

c \$200 500 invested of two-and-a-half years and earning \$30 075 interest

10 To buy his first car, Gary took out a personal loan for \$3500. He paid it back over a period of two years and this cost him \$1085 in interest. At what simple interest rate was he charged?

Calculating the time period of an investment or loan

11 If John invests \$20 000 at 10% per annum until he has \$32 000, for how many years will he have to invest the money?

12 How long will it take for \$17 000 invested at 15% per annum to grow to \$32 300?

13 Mikki decides to put \$6000 in the bank and leave it there until it doubles. If the money is earning simple interest at a rate of 11.5% per annum, how long will this take, to the nearest month?

14 Find the time taken for the following investments to earn the stated amounts of simple interest:

a \$2400 at 12% per annum earns \$175 interest

b \$700 at 4.9% per annum earns \$43 interest

Calculating the principal of an investment or loan

15 Over a period of five years an investment earned \$1070.25 at a simple interest rate of 5.45% per annum. What was the original amount deposited?

16 How much money needs to be invested in order to produce \$725 in interest calculated at 7.5% per annum simple interest over four years?

17 How much money needs to be invested at an interest rate of 3.5% per annum simple interest if you require \$10 000 in three years' time?

18 How long will it take, to the nearest month, for \$2200, invested at 12.75% per annum, to double in value?

Mixed problems

19 The local store advertises a stereo for \$1095, or \$100 deposit and \$32 per week for two years.

a How much does the stereo end up costing under this scheme?

b What equivalent rate of simple interest is being charged over the two years?

20 A personal loan of $10 000 over a period of three years is repaid at the rate of $400 per month.

- **a** How much money will be repaid in total?
- **b** What equivalent rate of simple interest is being charged over the three years?

21 Vicki invested $25 000 in bonds, which return monthly interest at the simple interest rate of 12.0% per annum.

- **a** What rate is paid each month?
- **b** How much interest does Vicki receive each month?
- **c** How much interest does Vicki receive each year?
- **d** How long does it take for the deposit to pay out $7500 interest?
- **e** How much interest does Vicki receive after a period of 10.5 years?

22 Maryanne invested $50 000 in a bank account that pays annual interest at the simple interest rate of 7% per annum.

- **a** Draw the graph of the interest earned each year against time (in years).
- **b** If the interest is paid into the same account, draw the graph of the amount in the account against time (in years).

20.3 Compound interest

Excel

Most interest calculations are not as straightforward as simple interest. The more usual form of interest is **compound interest**. It is called compound interest because the interest accumulated each year is added to the principal, and for each subsequent year interest is earned on this total of principal and interest. The interest thus compounds. Consider the situation where $5000 is invested at 10% interest per annum, and the interest is credited to the account annually:

Excel

In the first year:

$$\text{interest} = \$5000 \times 10\% \times 1 = \$500$$

and so at the end of the first year the amount of money in the account is:

$$\$5000 + \$500 = \$5500$$

Excel

In the second year:

$$\text{interest} = \$5500 \times 10\% \times 1 = \$550$$

and so at the end of the second year the amount of money in the account is:

$$\$5500 + \$550 = \$6050$$

PowerPoint

In the third year:

$$\text{interest} = \$6050 \times 10\% = \$605$$

and so at the end of the third year the amount of money in the account is:

$$\$6050 + \$605 = \$6655$$

The interest ($$I$) that would result from investing $$P$ at r% per annum, compounded annually for a time period of t years, is:

$$I = A - P = P \times \left(1 + \frac{r}{100}\right)^t - P$$

Example 11 Calculating the investment and interest with interest compounded annually

a Determine the amount of money accumulated after four years if $10 000 is invested at an interest rate of 9% per annum, compounded annually, giving your answer to the nearest dollar.

b Determine the amount of interest earned.

Solution

a Substitute $P = \$10\,000$, $t = 4$, $r = 9$ in the formula giving the amount of the investment.

$$A = P \times \left(1 + \frac{r}{100}\right)^t = 10\,000 \times \left(1 + \frac{9}{100}\right)^4$$
$$= 10\,000 \times 1.4116$$
$$= \$14\,116 \text{ to the nearest dollar}$$

b Subtract the principal from this amount to determine the interest earned.

$$I = A - P = 14\,116 - 10\,000$$
$$= \$4116$$

Another way of determining compound interest is to enter the appropriate formula into the graphics calculator, and examine the interest earned using the table and graph facilities of the calculator.

How to investigate compound interest problems using the TI-Nspire CAS

Determine the amount of money accumulated after 4 years if $10 000 is invested at an interest rate of 9% per annum, compounded annually. Give your answer to the nearest dollar.

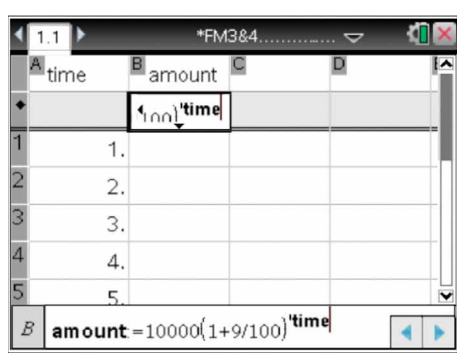
Steps

1 Substitute $P = 10\,000$, $r = 9$ in the compound interest formula.

$$A = 10\,000 \times \left(1 + \frac{9}{100}\right)^t$$

Method 1: Form a table

2 Start a new document by pressing (ctrl) + [N], and select **Add Lists & Spreadsheet**.
Name the lists *time* (to represent time in years) and *amount*.
Enter the data **1** to **10** into the list named *time*, as shown.



ISBN 978-1-107-65590-4

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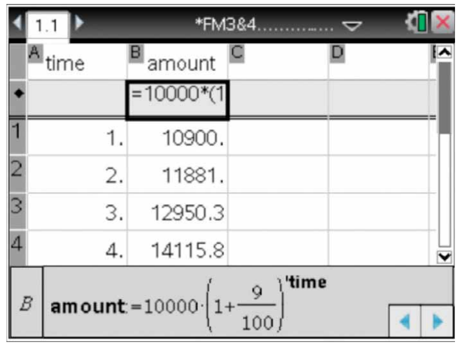
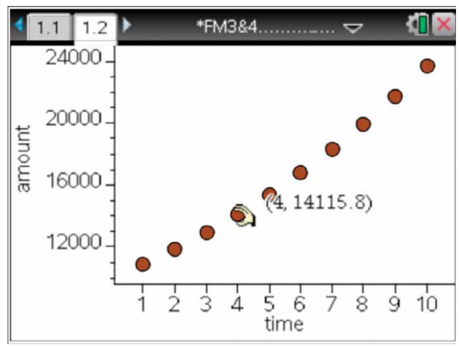
3 Place the cursor in the grey formula cell in the list named *amount* and type $= \mathbf{10\,000 \times (1 + 9 \div 100)^\wedge \; time}$. Press (enter) to display the values, as shown. Scrolling down the table we can see the amount of money accumulated after 4 years is \$14 116.



Method 2: Draw a graph

4 Press (ctrl) + [I] and select **Add Data & Statistics** and construct a scatterplot of *amount* against *time*, as shown.



Notes:

1. To connect the data points, press (ctrl) + (menu) and select **Connect Data Points**.
2. To display a value, place the cursor over the data point or use (menu) **>Analyze>Graph Trace**.
3. You can use (ctrl) + (menu) and select **Zoom>Window Settings** and set the **Ymin** to 0 if you prefer.

From the graph, we can see that the amount of money accumulated after 4 years is \$14 116.

How to investigate compound interest problems using the ClassPad

Determine the amount of money accumulated after four years if \$10 000 is invested at an interest rate of 9% per annum, compounded annually. Give your answer to the nearest dollar.

Steps

1 Substitute $P = 10\,000$, $r = 9$ in to the formula for compound interest.

$$A = 10\,000 \times \left(1 + \frac{9}{100}\right)^t$$

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Example 13 Calculating the interest on the investment or loan

Determine the interest earned in Example 12.

Solution

Subtract the principal from the amount of the investment to find the interest.

$$I = A - P = 3043.33 - 2700$$
$$= \$343.33$$

As was the case with simple interest, we often use the formula for compound interest to find the value of any of the variables in the equation when the values of the other variables are known. However, since the compound interest formula is quite complex, the easiest way to do this is to use the Equation Solver function of the graphics calculator.

How to solve for any variable in the compound interest formula using the TI-Nspire CAS

Suppose that an investment of \$2000 has grown to \$2123.40 after 12 months invested at r% per annum compound interest, compounded monthly. Find the value of r, correct to 1 decimal place.

Steps

1 The compound interest formula is

$$A = P \times \left[1 + \frac{r/n}{100}\right]^{nt}$$

Substitute $P = 2000$, $A = 2123.40$, $n = 12$ and $t = 1$ into this formula.
Use the **solve(..)** command to solve for r, the annual interest rate.

$$A = P \times \left(1 + \frac{r/n}{100}\right)^{nt}$$

$$2123.40 = 2000 \times \left(1 + \frac{r/12}{100}\right)^{12\times 1}$$

or

$$2123.40 = 2000 \times \left(1 + \frac{r}{1200}\right)^{12}$$

2 Start a new document by pressing ctrl + N.

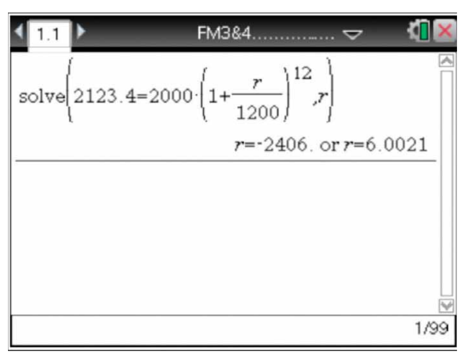
a Select **Add Calculator** and press menu **>Algebra>Solve** to paste in the **solve(..)** command.

b Complete the command by typing in the following equation to be solved and the unknown (r):

2123.40 = 2000 (1 + r ÷ 1200)^ 12, r

That is, **solve(2123.40 = 2000 (1 + r ÷ 1200)^ 12, r)**

Press enter to execute the command and display the answer.



Note: Use the ▶ arrow after typing in the **12** to return to the base line to finish typing the entry.

3 Note that there are two solutions, $r = -2406$ and $r = 6.002$. The interest rate cannot be negative, so $r = 6.0\%$.

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Example 14 Using the solve command to find the initial investment

How much money must you deposit at 7% per annum compound interest, compounding yearly, if you require $10 000 in three years' time? Give your answer to the nearest dollar.

Solution

The compound interest formula is

$$A = P \times \left(1 + \frac{r/n}{100}\right)^{nt}$$

$$A = P \times \left(1 + \frac{r/n}{100}\right)^{nt}$$

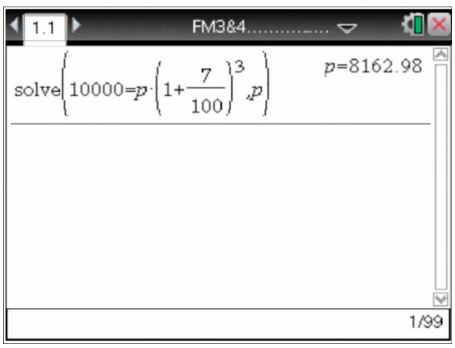
1 Substitute $A = 10\,000$, $r = 7$, $n = 1$ and $t = 3$ into this formula.

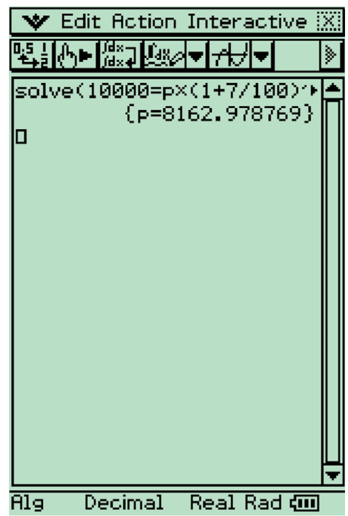
$$10\,000 = P \times \left(1 + \frac{7/1}{100}\right)^{1\times 3}$$

or

$$10\,000 = P \times \left(1 + \frac{7}{100}\right)^{3}$$

2 Use the **solve(** command to solve for P (the principal).





3 Write the answer.

Answer: Deposit $8163

Example 15 Using the solve command to find a time period

How long, to the nearest year, will it take for an investment of $1000 to reach $1873 if it is invested at 9% per annum compounded monthly?

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Solution

The compound interest formula is

$$A = P \times \left(1 + \frac{r/n}{100}\right)^{nt}$$

$$A = P \times \left(1 + \frac{r/n}{100}\right)^{nt}$$

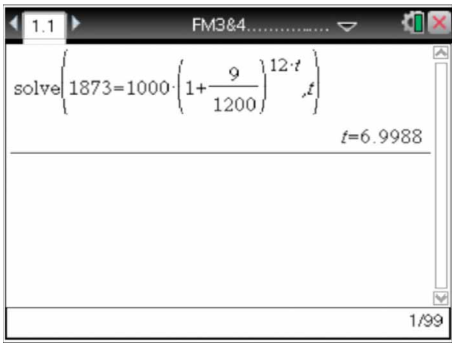
1 Substitute $A = 1873$, $P = 1000$, $r = 9$, and $n = 12$ into the formula.

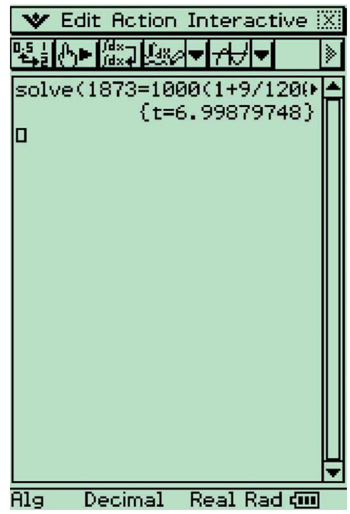
$$1873 = 1000 \times \left(1 + \frac{9/12}{100}\right)^{12 \times t}$$

or

$$1873 = 1000 \times \left(1 + \frac{9}{1200}\right)^{12t}$$

2 Use the **solve(** command to solve for t.





Hint for ClassPad:
When entering the $12t$ term, place brackets around it $(12t)$; hence, enter the expression as
solve(1873 = 1000 × (1 + 9/1200)^(12t))

3 Write the answer.

Answer: Invest for 7 years.

Exercise 20C

Skillsheet TEACHER

1 Calculate the compound interest for the following:
  a \$2000 invested at 6% per annum for four years
  b \$10 000 invested at 12% per annum for five years
  c \$8000 invested at 12.5% per annum for three years

2 How much money would be in an account after five years if \$3000 is invested at 10% per annum compounded annually?

3 How much interest is earned if \$3300 is invested for 10 years at 7.5% per annum compounded annually?

4 Compare the interest earned with both simple and compound interest if \$4500 is invested at 11% per annum for six years. What is the amount of the difference?

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Example 6 **Finding the interest earned**

Andrea receives a statement from the bank which gives the detail of her investment account from 1 July until 31 December, 2005. The details are as shown.

Date	*Debit ($)*	*Credit ($)*	*Total($)*
Opening Balance:			
1 July			2000.00
8 August		360.00	2360.00
10 September	1363.40		996.60
Closing Balance:			
31 December			996.60

How much interest has been earned on this account if the bank pays simple interest of 4.5% per annum on the minimum daily balance?

Solution

1 The balance from 1 July until 8 August is $2000. How many days is this?

1 July until 8 August $= 31 + 8 = 39$ days

2 Determine the interest payable for this period.

$$\text{Interest} = 2000 \times \frac{4.5}{100} \times \frac{39}{365} = 9.616$$

3 The balance from 9 August until 10 September is $2360. How many days is this?

9 August until 10 September $= 23 + 10 = 33$ days

4 Determine the interest payable for this period.

$$\text{Interest} = 2360 \times \frac{4.5}{100} \times \frac{33}{365} = 9.602$$

5 The balance from 11 September until 31 December is $3723.40. How many days is this?

11 September until 31 December
$= 20 + 31 + 30 + 31 = 112$ days

6 Determine the interest payable for this period.

$$\text{Interest} = 996.60 \times \frac{4.5}{100} \times \frac{112}{365} = 13.761$$

7 Determine the total interest.

Total interest $= 9.616 + 9.602 + 13.761 = 32.979$
Thus the total interest earned is $32.98.

Exercise 21B

1 An account at a bank is paid interest of 0.75% per month on the minimum monthly balance, credited to the account at the beginning of the next month. During a particular month the following transactions took place:

7 September	$1500 withdrawn
12 September	$950 withdrawn

7 The details of an investment account are as shown.

How much interest has been earned on this account if the bank pays simple interest of 3% per annum on the minimum daily balance?

Date	*Debit ($)*	*Credit ($)*	*Total ($)*
Opening Balance:			
1 July			500.00
8 August		400.00	900.00
10 December		350.00	1250.00
Closing Balance:			
31 December			1250.00

8 Andrew receives a statement from the bank that gives the detail of his investment account from 1 November until 31 December. The details are as shown.

How much interest has been earned on this account if the bank pays simple interest of 4% per annum on the minimum daily balance?

Date	*Debit ($)*	*Credit ($)*	*Total ($)*
Opening Balance:			
1 November			10 000.00
12 November		4350.98	14 350.98
11 December	2277.44		12 073.54
Closing Balance:			
31 December			

21.3 Time payments (Hire purchase)

Instead of saving for the purchase of an item, an option is to enter into a hire-purchase agreement. This means the purchaser agrees to hire the item from the vendor and make periodical payments at an agreed rate of interest. At the end of the period of the agreement, the item is owned by the purchaser. If the purchaser stops making payments at any stage of the agreement, the item is returned to the vendor and no money is refunded to the purchaser.

The interest rate being charged in these contracts is not always stated explicitly. There are two different interest rates that could be stated, or determined, and it is important to distinguish between them so that we can judge just how much we are paying for an item. These are the flat rate of interest, r_f, and the effective rate of interest, r_e.

Flat rate of interest

The interest paid given as a percentage of the original amount owed is called the **flat rate of interest**. The flat interest rate is exactly the same as the simple interest rate, but is generally called by this name in the hire-purchase context. In Chapter 20 we established that for simple interest:

$$r = \frac{100I}{Pt}$$

where P is the principal, I is the amount of interest, t is the time in years.

How to determine flat rate depreciation and book value using the TI-Nspire CAS

A computer system costs \$9500 to buy and decreases in value by 10% of the purchase price each year.

a What is the amount of depreciation after 4 years?

b Find its book value after 4 years.

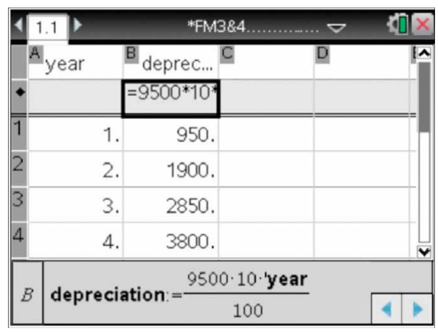
Steps

1 Write expressions for depreciation and book value.

$$\text{depreciation} = \frac{9500 \times 10 \times t}{100}$$

$$\text{book value} = P - \frac{Prt}{100} = 9500 - \frac{9500 \times 10 \times t}{100}$$

2 Start a new document by pressing (ctrl) + [N], and select **Add Lists & Spreadsheet**.



a Name three lists: *year* (for *t*), *depreciation*, and *book_value*, respectively.
Hint: Use (ctrl) + [␣] for the underscore or just type *bookvalue*.

b Enter the numbers **1, 2, 3, . . . , 10** into the list *year*.

c Move the cursor to the grey formula cell of the list *depreciation* and type

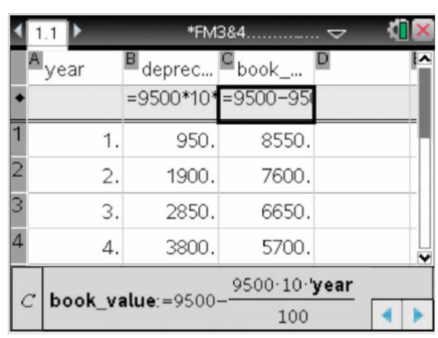
$= \mathbf{(9500 \times 10 \times \textit{year})/100}$

Press (enter) to calculate the values for depreciation.

3 Move the cursor to the grey formula cell of the list *book_value* and type

$= \mathbf{9500 - (9500 \times 10 \times \textit{year})/100}$

Press (enter) to calculate the values for book value.



Notes:

1 An alternative formula to use to calculate the list *book_value* would be

$= \mathbf{9500} - \textbf{\textit{depreciation}}$

2 You can use the (var) key to display the variable list rather than retyping the list names.

4 Scrolling through the table we see that after 4 years the depreciation is \$3800 and the book value is \$5700.

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How to determine reducing balance depreciation and book value using the TI-Nspire CAS

A computer system costs \$9500 to buy and decreases in value by 20% each year.

a What is the book value of the computer after four years?

b By how much has the value of the computer depreciated over the four years?

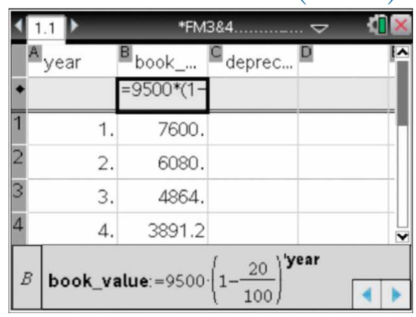
Steps

1 Write expressions for book value and depreciation.

$$\text{book value} = 9500 \times \left(1 - \frac{20}{100}\right)^t$$

$$\text{depreciation} = 9500 - 9500 \times \left(1 - \frac{20}{100}\right)^t$$

2 Start a new document by pressing ctrl + N, and select **Add Lists & Spreadsheet**.



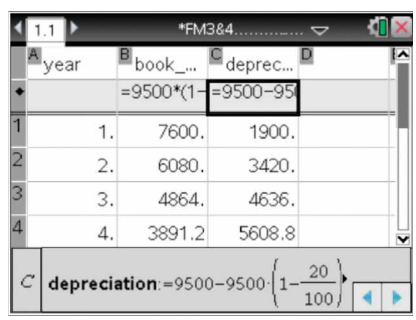
a Name three lists, *year* (t), *book_value* and *depreciation*, respectively.
Hint: Use ctrl + ␣ for the underscore or just write as *bookvalue*.

b Enter the numbers **1**, **2**, **3**, . . . , **10** into the list *year*.

c Move the cursor to the grey formula cell of the list *book_value* and type
= (9500 × (1 − 20/100)^*year*
Press enter to calculate the values for book value.

3 Move the cursor to the grey formula cell of the list *depreciation* and type
= 9500 − (9500 × (1 − 20/100)^*year*
Press enter to calculate the values for depreciation.

Note: You can use the var key to display the variable list rather than retyping in the list names.



4 Scrolling through the table, we see that after four years the book value is \$3891.20 and the depreciation is \$5608.80.

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Exercise 21E

Unit cost depreciation

1 A machine originally costing $37 000 is expected to produce 100 000 units. The output of the machine in each of the first three years was 5234, 6286 and 3987 units respectively. Its anticipated scrap value is $5000.

a What is the unit cost for this machine?

b Find the total production over the first three years, and hence the book value at the end of three years.

c Estimate how many years it will be in use, if the average production during its life is 5169 units per year.

2 A company buys a taxi for $29 000. It depreciates at a rate of 25 cents per kilometre. If the taxi has a scrap value of $5000, find how many kilometres it will have travelled by the time it reaches its scrap value.

3 If a car is valued at $35 400 at the start of the year, and at $25 700 at the end of the year, what has been the unit cost per kilometre if it travelled 25 000 km that year?

4 A printing machine costing $110 000 has a scrap value of $2500 after it has printed 4 million pages.

a Find:

i the unit cost of the machine

ii the book value of the machine after printing 1.5 million pages

iii the annual depreciation charge of the machine if it prints 750 000 pages per year

b Find the book value of the printing machine after five years if it prints, on average, 750 000 pages per year.

c How many pages has the machine printed by the time the book value is $70 000, if it prints, on average, 750 000 pages per year?

Flat rate depreciation

5 A sewing machine originally cost $1700 and decreases in value by 12.5% of the purchase price per year.

a What is the amount of the depreciation after 3 years?

b Find its book value after 3 years.

6 A harvester was bought for \$65 000 and it decreases by 10% of the purchase price per annum.

a Write down the rule relating book value, flat rate of depreciation and time in years.
b Use your graphic calculator to draw a graph of book value against time in years.
c Find the amount of the yearly depreciation.
d If the scrap value of the harvester is \$13 000, for how many years will it be in use?

7 A computer depreciates at a flat rate of 22.5% of the purchase price per annum. Its purchase price is \$5600.

a What is the book value of the computer after 3 years?
b After how long will the computer be written off if the scrap value is nil?

8 A machine costs \$7000 new and depreciates at a flat rate of 17.5% per annum. If its scrap value is \$875, find:

a the book value of the machine after two years
b after how many years the machine will be written off

Reducing balance depreciation

9 A stereo system purchased for \$1200 incurs 12% per annum reducing balance depreciation.

a Find the book value after 7 years.
b What is the total depreciation after 7 years?
c If the stereo has a scrap value of \$215, in which year will this value be reached?

10 A car costing \$38 500 depreciates at a rate of 9.5% per year. Give your answers to the following to the nearest dollar.

a What is the book value of the car at the end of five years?
b What is the total amount of depreciation after five years?
c If the car has a scrap value of \$10 000, in which year will this value be reached?

11 A machine has a book value after 10 years of \$13 770. If it has depreciated at a reducing balance rate of 8.2% per annum, what was the initial cost of the machine?

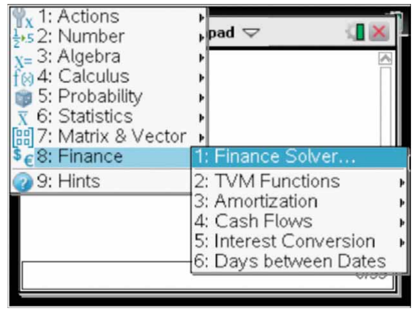
12 After depreciating at a reducing balance rate of 12.5% per annum, a yacht is now worth \$56 100. What was the yacht worth when it was new six years ago, to the nearest \$100?

13 What reducing balance rate would cause the value of a car to drop from \$8000 to \$6645 in three years?

How to use Finance Solver on the TI-Nspire CAS

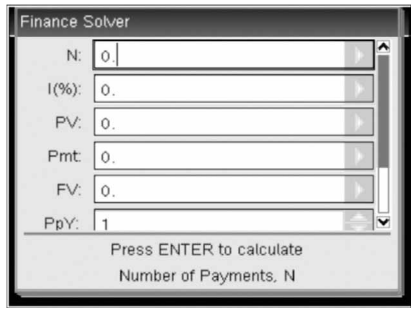
Steps

1 Press (⌂ on) (or (on) then (⌂) on the Clickpad). Then press [A] to open the **Scratchpad: Calculate**.



2 Press (menu) **>Finance>Finance Solver**.

3 To use **Finance Solver** you need to know the meaning of each of its symbols. These are as follows:



- **N** is the total number of payments
- **I(%)** is the annual interest rate
- **PV** is the present value of the loan or investment
- **Pmt** is the amount paid at each payment
- **FV** is the future value of the loan or investment
- **PpY** is the number of payments per year
- **CpY** is the number of times the interest is compounded per year. (It is almost always the same as **PpY**.)
- **PmtAt** is used to indicate whether the interest is compounded at the end or at the beginning of the time period. Leave this set at **END**.

Note: Use (tab) or ▼ to move down boxes. Press ▲ to move up. For **PpY** and **CpY** press (tab) to move down to the next entry box.

4 When using **Finance Solver** to solve loan and investment problems, there will be one unknown quantity. To find its value, move the cursor to its entry box and press (enter) to solve.

Now we can consider each of the applications of the finance solver separately.

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Reducing balance loans

Reducing balance loans were introduced in Section 20.4. Essentially, this describes a situation where a loan is taken out under compound interest, and period repayments are made.

Example 17 **Determining the repayment amount, total cost and total amount of interest paid**

Simone borrows $10 000 to be repaid over a period of 5 years. Interest is charged at the rate of 8% per annum compounding monthly. Find:

a the monthly repayment correct to the nearest cent

b the total cost of paying off the loan, to the nearest dollar

c the total amount of interest paid

Solution

a **1** Open the finance solver on your calculator and enter the information below, as shown opposite.

- **N**: 60 (number of monthly payments in 5 years)
- **I%**: 8 (annual interest rate)
- **PV**: 10 000 (positive as this is the amount the bank has given to you)
- **Pmt** or **PMT** (the unknown payment): leave blank or **Clear**
- **FV**: 0 (loan is to be paid off completely)
- **Pp/Y**: 12 (monthly payments)
- **Cp/Y**: 12 (interest compounds monthly)

N:	60
I%:	8
PV:	10000
Pmt or PMT:	
FV:	0
Pp/y or P/Y:	12
Cp/y or C/Y:	12

2 Solve for the unknown. On the:

TI-Nspire: Move the cursor to the **PMT** entry box and press (enter) to solve.

ClassPad: Tap on the **PMT** box to the left.

The amount –202.7639 . . . now appears in the **Pmt or PMT** entry box.

Note: The sign of the payment is negative because it is the amount that Simone must give back (repay) to the bank each month.

N:	60
I%:	8
PV:	10000
Pmt or PMT:	–202.7639…
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

Simone must repay the bank $202.76 per month.

b total cost of paying off the loan $=$ total number of payments $\times$ payment amount

total cost $= 5 \times 12 \times 202.76 =$ *$12 166 (to the nearest dollar)*

c total interest paid $=$ total cost of paying off the loan – amount of loan

total interest paid $= 12\,166 - 10\,000 =$ *$2166*

Using a finance solver we can solve for any of the variables listed.

Example 18 Determining the amount owed and the number of repayments for a reducing balance loan

Andrew borrows $20 000 at 7.25% per annum, compounded monthly, and makes monthly repayments of $200.

a How much does he owe after three years?

b How long will it take him to pay out the loan? Give your answer to the nearest month.

Solution

a **1** Open the finance solver on your calculator and enter the information below, as shown opposite.

- **N**: 36 (number of monthly payments in 3 years)
- **I%**: 7.25 (annual interest rate)
- **PV**: 20 000 (positive, as this is the amount the bank has given to you)
- **Pmt or PMT**: –200 (negative, as this is the amount you give back to the bank each month)
- **FV** (the unknown quantity): leave blank or **Clear**
- **Pp/Y**: 12 (monthly payments per year)
- **Cp/Y**: 12 (interest compounds monthly)

N:	36
I%:	7.25
PV:	20000
Pmt or PMT:	–200
FV:	
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Solve for the unknown. On the:

TI-Nspire: Move the cursor to the **FV** entry box and press (enter) to solve.

ClassPad: Tap on the **FV** box on the left.

The amount –16 826.97 . . . now appears in the **FV** entry box.

Note: The sign of the future value is negative, indicating that this is money that is still owed to the bank and must, eventually, be paid back. See page 591.

N:	36
I%:	7.25
PV:	20000
Pmt or PMT:	–200
FV:	–16826.97…
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

3 Write your answer.

After three years, Andrew still owes the bank $16 826.97

b **1** **N**, the total number of payments, is now the unknown. Leave blank or **Clear**.

Change **FV** to **0** to indicate the loan is to be fully paid out.

All other values stay the same.

N:	
I%:	7.25
PV:	20000
Pmt or PMT:	–200
FV:	0
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

Example 22 **Interest only loans**

Jane borrows $500 000 to buy shares. If the interest on the loan is 6.65% per annum, compounding monthly, what will be her monthly repayment on an interest only loan?

Solution

1 We will consider the situation for one year only; all other years will be the same. Using your finance solver, solve for **Pmt** with:

- **N:** 12
- **I%:** 6.65
- **PV:** 500 000
- **FV:** −500 000 (negative, as Jane will eventually have to pay this money back to the lender)
- **Pp/Y:** 12
- **Cp/Y:** 12

Note: Pmt or PMT will be negative as this amount will need to be paid back to the lender at the end of each month.

N:	12
I%:	6.65
PV:	500000
Pmt or PMT:	–2770.83...
FV:	–500000
Pp/y or P/Y:	12
Cp/Y or C/Y:	12

2 Write your answer.

Jane's monthly repayments are $2770.83.

The repayment on the interest only loan is equivalent to paying only the simple interest due on the principal for one year. This can be readily verified using the information in Example 22.

Perpetuities

A perpetuity is an investment that pays out an equal amount, hopefully forever! For example, you might want to start a scholarship at your school where, every year, a student will receive $1000. You want this scholarship to continue indefinitely, even after you are long gone. The question is, how much money will it cost you? This is just an application of simple interest.

If we invest P dollars for one year at an interest rate of r% per annum, then, at the end of the year, the amount, A, that we will have in our investment fund is given by:

$$A = \text{amount invested} + \text{interest earned} = P + P \times \frac{r}{100}$$

If we only spend the interest earned at the end each year, then we will always have the original P dollars to reinvest each year and always be able to make a regular payment of $Q = P \times \dfrac{r}{100}$ dollars each year.

In a perpetuity, if an amount, $\$P$, is invested at an interest rate of r% per annum, and a regular payment of $\$Q$ per annum is made, then:

$$Q = \frac{Pr}{100} \qquad \text{or} \qquad P = \frac{100Q}{r}$$

Example 23 **Perpetuity investment amount**

Suppose that Richard wishes to start a scholarship, where every year a student receives $1000. If the interest on the initial investment averages 3% per annum, how much should be invested?

Solution

Substitute $Q = \$1000$ and $r = 3$ into the formula for a perpetuity.

$$P = \frac{100Q}{r} = \frac{100 \times 1000}{3} = 33\,333.33$$

Answer: Invest $33 333.33

Example 24 **Perpetuity payment**

Elizabeth places her superannuation payout of $500 000 in a perpetuity that will provide a monthly income without using any of the principal. If the interest rate on the perpetuity is 6% per annum, what monthly payment will Elizabeth receive?

Solution

1 To determine how much Elizabeth will receive per annum substitute $P = \$500\,000$ and $r = 6$ into the formula to find the payment, Q.

$$Q = \frac{Pr}{100} = \frac{500000 \times 6}{100}$$
$$= \$30\,000$$

2 Since this is the annual payment, we divide by 12 to find the monthly payment.

$$\text{Monthly payment} = \frac{30\,000}{12} = \$2500$$

Exercise 21F

Reducing balance loans

1 A loan of $90 000 is to be repaid over a period of 30 years. Interest is charged at the rate of 11% per annum compounding monthly. Find:

- **a** the monthly repayment correct to the nearest cent
- **b** the total cost of paying off the loan to the nearest dollar
- **c** the total amount of interest paid

2 A building society offers $240 000 home loans at an interest rate of 10.25% compounding monthly.

- **a** If repayments are $2200 per month, calculate the amount still owing on the loan after 12 years. Give your answer correct to the nearest dollar.
- **b** If the loan is to be fully repaid after 12 years, calculate:
  - **i** the monthly repayment, correct to the nearest cent
  - **ii** the total amount repaid, correct to the nearest dollar
  - **iii** the total amount of interest paid

3 A loan of \$10 000 is to be repaid over 5 years. Interest is charged at the rate of 11% per annum compounding quarterly. Find:

a the quarterly repayment, correct to the nearest cent
b the total cost of paying off the loan, to the nearest dollar
c the total amount of interest paid

4 Interest on a reducing balance loan of \$65 000 is compounded quarterly at an interest rate of 12.75% per annum. Calculate the quarterly repayment if:

a the amount still owing after 10 years is \$25 000
b the amount still owing after 20 years is \$25 000
c the loan is fully repaid after 10 years
d the loan is fully repaid after 20 years

5 Dan arranges to make repayments of \$450 per month to repay a loan of \$20 000, with interest being charged at 9.5% per annum compounded monthly. Find:

a the number of monthly repayments required to pay out the loan (to the nearest month)
b the amount of interest charged

6 Joan considers taking out a loan on the terms given in Question 5. However, she decides that she can afford higher monthly repayments of \$550.

a How long does it take her to pay off her loan (to the nearest month)?
b How much interest does Joan save by paying the higher monthly instalment?

7 A loan of \$600 000 is taken out to finance a new business. The loan is to be repaid fully over 10 years with quarterly payments of \$23 690.90.
Determine the annual interest rate for this loan. Give your answer correct to two decimal places.

Annuities

8 Stephanie purchases a \$40 000 annuity, with interest paid at 7.5% per annum compounded monthly. If she wishes to receive a monthly payment for 10 years, how much will she receive each month?

9 Lee purchases an annuity for \$140 000, with interest of 6.25% per annum compounded monthly. If he receives payments of \$975 per month, how long will the annuity last? Give your answer to the nearest month.

10 Raj purchases an annuity for \$85 500, with interest of 7.25% per annum compounded quarterly.

a If he receives quarterly payments for 10 years, how much will he receive each quarter?
b If he receives a regular quarterly payment of \$5000, how long will the annuity last? Give your answer to the nearest quarter.

Adding to an investment

11 Bree has \$25 000 in an account that pays interest at a rate of 6.15% per annum compounding monthly.

- **a** If she makes monthly deposits of \$120 to the account, how much will she have in the account at the end of 5 years?
- **b** If she makes monthly withdrawals of \$120 from the account, how much will she have in the account at the end of 5 years?

12 Jarrod saves \$500 per month in an account that pays interest at a rate of 6% per annum compounding monthly.

- **a** If he makes monthly deposits of \$500 to the account, how much will he have in the account at the end of 10 years?
- **b** Suppose that, after 10 years of making deposits, Jarrod starts withdrawing \$500 each month from the account. How much will he have in the account at the end of another 10 years?

Comparing loans

13 Mr and Mrs Kostas decide to borrow \$25 000 to help them finance the construction of their swimming pool. They consider two loan repayment options:
Loan option A: Monthly repayments of 7.5% per annum compounded monthly
Loan option B: Quarterly repayments at 7.5% per annum compounded quarterly
They wish to pay off the loan over 5 years. Calculate, to the nearest dollar, for each loan:

- **a** the total repayment
- **b** the total interest paid and hence decide which, if either, is the better loan

14 If Mr and Mrs Kostas of Question 13 choose Loan option A, how much interest do they pay if the interest rate is increased by 0.5%?

15 A flat rate loan over 6 years at 12.75% per annum amounted to a repayment of \$12 500.

- **a** How much was originally borrowed?
- **b** Calculate the quarterly repayments.
- **c** Compare the savings of a reducing balance loan by working out the quarterly repayments with interest set at 12.75% per annum compounded quarterly.
- **d** How much is saved over the full 6-year period by adopting a reducing balance loan?

16 A personal loan of \$7500 is taken out at 11.5% per annum over 4 years.

- **a** Calculate the total amount to be repaid:
  - **i** if the loan was a flat rate loan
  - **ii** if the loan was a reducing balance loan with monthly repayments
- **b** What flat rate payment of interest would the monthly repayment in **a** part **ii** be equivalent to?

17 A credit institution offers loans of up to \$12 000 at an interest rate of $9\frac{1}{3}$% per annum calculated on the principal.

a Calculate the interest charged on a loan of \$9500 after three years.

b Calculate the monthly repayments on the loan in **a** if it is to be fully repaid in five years.

c If monthly repayments of \$370.75 are made, how long does it take to fully repay the loan?

Interest only loans

18 Georgia borrows \$100 000 to buy an investment property. If the interest on the loan is 7.15% per annum compounding monthly, what will be her monthly repayment on an interest only loan?

19 In order to invest in the stockmarket, Jamie takes out an interest only loan of \$50 000. If the interest on the loan is 8.15% per annum compounding monthly, what will be his monthly repayments?

20 Jackson takes out an interest only loan of \$30 000 from the bank to buy a painting, which he hopes to resell at a profit in 12 months' time. The interest on the loan is 9.25% per annum compounding monthly, and he makes monthly payments on the loan. How much will he need to sell the painting for in order not to lose money?

Perpetuities

21 Geoff wishes to set up a fund so that every year \$2500 is donated to the RSPCA in his name. If the interest on his initial investment averages 2.5% per annum, compounded annually, how much should he invest?

22 Barbara wishes to start a scholarship that will reward the top mathematics student each year with a \$500 prize. If the interest on the initial investment averages 2.7% per annum compounded annually, how much should be invested? Give your answer to the nearest dollar.

23 Cathy wishes to maintain an ongoing donation of \$5500 per year to the Collingwood Football Club. If the interest on the initial investment averages 2.75% per annum compounded annually, how much should she invest?

24 Craig wins \$1 000 000 in a lottery and decides to place it in a perpetuity that pays 5.75% per annum interest compounding monthly. What monthly payment does he receive?

25 Suzie invests her inheritance of \$642 000 in a perpetuity that pays 6.1% per annum compounding quarterly. What quarterly payment does she receive? After five quarterly payments, how much money remains invested in the perpetuity?

Reducing balance loan with changing conditions

26 An amount of $35 000 is borrowed for 20 years at 10.5% per annum compounded monthly.

a What are the repayments for the loan?
b How much interest is paid on the loan over the 20-year period?
c How much is still owing at the end of 4 years?

After four years, the interest rate rises to 13.75% per annum.

d What are the new repayments that will see the amount repaid in a total of 20 years?
e How much extra must now be repaid on the loan over the term of 20 years?

27 A couple negotiates a 25-year mortgage of $150 000 at a fixed rate of 7.5% per annum compounded monthly for the first 7 years, then at the market rate for the remainder of the loan. They agree to monthly repayments of $1100 for the first 7 years. Calculate:

a the amount still owing after the first 7 years
b the new monthly repayments required to pay off the loan if after 7 years the market rate has risen to 8.5% per annum

28 A couple puts a $20 000 down-payment on a new home and arranges to pay off the rest in monthly instalments of $625 for 30 years at a monthly compounded interest rate of 8.5% per annum.

a What was the selling price of the house, to the nearest cent?
b How much interest will they pay over the term of the loan?
c How much do they owe after 6 years?

After 6 years the interest rates increase by 0.9%. The couple must now extend the period of their loan in order to pay it back in full.

d How much will they still owe after the original 30-year period?
e Will they ever repay the loan at their original monthly repayment of $625?
f Calculate the new monthly repayment amount required if the couple still wishes to pay off the loan in 30 years.

MC TEST D&D TEST

Revision

26 Peter borrows \$80 000 for 10 years at 5.6% per annum, compounding monthly, with monthly repayments of \$555. Which one of the following statements is true?

A The loan will be fully paid out in 10 years.
B At the end of 5 years, the balance of the loan will be \$40 000.
C The amount of interest paid each month during the loan increases.
D Weekly payments of \$132 compounding weekly would reduce the period of the loan.
E If one extra payment of \$2000 is to be made, it would be better to make it at the end of year 8 than at the end of year 2. [VCAA pre 2006]

27 At the start of each year, Joe's salary increases to take inflation into account.
Inflation averaged 2% per annum last year and 3% per annum the year before that.
Joe's salary this year is \$42 000.
Joe's salary two years ago, correct to the nearest dollar, would have been:

A \$39 900 **B** \$39 925 **C** \$39 925 **D** \$39 976 **E** \$39 977 [VCAA 2007]

28 Alf and Rani each invest \$2500 for five years.
Alf's investment earns simple interest at the rate of 8.5% per annum.
Rani's investment earns interest at the rate of 7.25% per annum compounding monthly.
After 5 years, correct to the nearest dollar, Alf will have:

A \$26 less than Rani **B** \$26 more than Rani
C \$16 less than Rani **D** \$16 more than Rani
E the same amount of money as Rani

29 \$5000 is invested at a rate of r% per annum compounding quarterly. The value, in dollars, of this investment after two-and-a-half years is given by:

A $5000\left(1+\dfrac{r}{100}\right)^{10}$ **B** $5000\left(1+\dfrac{r}{400}\right)^{2.5}$ **C** $5000\left(1+\dfrac{r}{100}\right)^{2.5}$

D $5000\left(1+\dfrac{r}{1200}\right)^{2.5}$ **E** $5000\left(1+\dfrac{r}{400}\right)^{10}$

22.2 Extended-response questions

1 Adele decides to spend her money as follows:
\$40 000 on a new car
\$40 000 on the latest computer equipment
Adele knows that the car will depreciate by 25% per annum based on the reducing value of the car, whereas the computer equipment will depreciate at a flat rate of \$8000 per year.

a What is the value of the car after:
i one year? **ii** three years?
b What is the value of the computer equipment after two years?
c After how many full years does the depreciated value of the car first exceed the depreciated value of the computer equipment?
d Determine the annual percentage flat rate depreciation applied to the computer equipment. [VCAA pre 2006]

2 Sally's credit union passbook looked like this in June 2012.

Date	*Particulars*	*Deposits*	*Withdrawals*	*Balance*
01 July 2011	Brought forward			2400.00
15 Dec 2011	Deposit	1200.00		3600.00
02 Feb 2012	ATM withdrawal			3000.00
14 May 2012	Interest	85.50		
20 June 2012	ATM withdrawal		450.00	2635.50

a What was:

i the amount withdrawn on 2 February 2012?

ii the account balance for 14 May 2012?

b Interest on this account was paid at a rate of 0.3% per month, based on a minimum monthly balance. How much interest did Sally earn for the month of December 2011? [based on VCAA pre 2006]

3 On 1 July 2012, Sally invested \$4000 in a new term deposit that offered a total of \$416 interest after two years.

a What was the annual simple interest rate offered for this term deposit?

b An alternative option for Sally had been to invest with a bank at a rate of 4.8% per annum compounding annually. To calculate the total amount in this account after two years with this option, Sally wrote down an equation that looked like this:

$$\text{total amount} = 4000 \times c \times c$$

What number should Sally have used for c?

c What annual compounding interest rate, correct to two decimal places, would Sally have needed to earn \$416 interest in two years on a \$4000 investment?

[based on VCAA pre 2006]

4 Lucy wants to borrow \$25 000. Interest is calculated quarterly on the reducing balance at an interest rate of 7.9%.

a If Lucy can afford to repay her loan at \$1600 per quarter:

i How much of Lucy's first payment is interest?

ii Will repayments of \$1600 enable Lucy to repay the loan within four years? Explain.

b Suppose Lucy arranges to pay \$1525 per quarter.

i How long will it take her to pay back the loan? Give your answer to the nearest quarter.

ii How much will the period of Lucy's loan be reduced if her payments are increased to \$1745? Give your answer to the nearest quarter.

5 Eric wants to buy a photocopier. Crazy Bob's normally sells them for \$4450, but they have a special discounted price of \$3800 for this week.

a What is the percentage discount? Write your answer correct to one decimal place.

b Crazy Bob's offers to sell the photocopier for the discounted price of \$3800, with terms of \$500 deposit and \$330 per month for 12 months.

(*cont'd.*)

i What is the total cost of the photocopier on these terms?
ii What is the annual flat rate of interest charged?

c Eric sees the same photocopier for sale at Discount House, also for \$3800. The terms of the sale there require no deposit and monthly repayments over two years at an interest rate of 8.5% per annum, calculated monthly on the reducing balance.
i What is the monthly repayment for this loan? Write your answer in dollars correct to two decimal places.
ii What is the total cost of the machine from Discount House on these terms? Write your answer correct to the nearest dollar.

d Whose terms, Crazy Bob's or Discount House, offer the lowest total cost for the photocopier? Justify your answer by calculating the difference in total money paid.

6 Brad buys a coffee machine for his café with an initial value of \$3100. He considers two methods of depreciating the value of the coffee machine.

a Suppose that the value of the machine is depreciated using the reducing balance method over 3 years and reducing at a rate of 15% per annum. What is the depreciated value of the machine after 3 years? Write your answer correct to the nearest dollar.

b Alternatively, suppose that the machine is depreciated using the unit cost method. Brad sells 15 000 cups of coffee per year and the unit cost per cup is 3.0 cents. Determine the depreciated value of the machine after 3 years. Write your answer correct to the nearest dollar.

c Brad wants the depreciated value of the machine after 3 years to be the same when calculated by both methods of depreciation. What would the unit cost per cup of coffee have to be for this to occur? Write your answer in cents, correct to one decimal place. [VCAA pre 2006]

7 Roslyn earns an annual salary of \$54 200, which is paid monthly. She did not join the superannuation fund until her 37th birthday and she now pays 7% of her gross salary to the superannuation fund. Her employer contributes a further 14%.

a What amount of money is placed each month into her superannuation fund?

b The superannuation fund pays 4.2% per annum compound interest, compounded monthly. Assuming that Roslyn's annual salary remains constant, what is the amount of superannuation she will have available at her 60th birthday?

c If there is an average of 2.5% inflation over the period of time that Roslyn is working, what is the purchasing power of the amount of superannuation determined in part **b**?

d Suppose that when Roslyn retires she places her superannuation in a perpetuity that will provide a monthly income without using any of the principal. If the perpetuity pays 4.25% per annum compounding monthly, what monthly payment will Roslyn receive?

8 Shelly decides to sell her business and invest the proceeds in an investment account that pays 5.5% per annum interest, compounding monthly. She plans to continue to work for five more years and add another \$1500 per month to the account, and then retire.

a If she makes a profit of $825 000 on her business, how much will Shelly have in the investment account when she retires?

b If there has been an average inflation rate of 3.2% over the 5-year period of her investment, what is the purchasing power of the amount of money Shelly has in her account?

c When she retires in five years, Shelly plans to use her money to buy an annuity, which pays 5.75% per annum compounding monthly. If she receives $8400 per month for her living expenses, how long will the annuity last?

d Alternatively, Shelly could place the money in a perpetuity. If the perpetuity she selects pays 5.75% per annum compounding monthly, how much is the monthly payment that Shelly will receive?

9 Glenda decides to buy a house worth $250 000 with a deposit of $80 000, and a loan of $170 000 from a building society. To repay the loan of $170 000, Glenda pays the building society $1850 per month for 10 years.

a Calculate the total amount of Glenda's repayments to the building society.

b Determine the total interest on the loan during the 10 years.

c Find the annual flat rate of interest charged by the building society. Give your answer correct to one decimal place.

10 Robyn invests $100 000 to provide a scholarship valued at $10 000 to the best mathematics student in the final year at her old school. She invests the money into an annuity at an interest rate of 8.25% per annum compound interest. She makes the payment to the winning student each year immediately after the interest is paid into the account.

a How much money is left in the account after the first two scholarships are awarded?

b Determine the amount that is left in Robyn's account after 10 years of awarding scholarships. Give your answer to the nearest cent.

c What would be the maximum value for each scholarship if they are to be awarded forever?

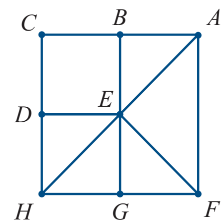
d How much would Robyn need to invest to be able to pay the $10 000 scholarship in perpetuity? Give your answer to the nearest dollar.

It is easy to remember the difference between Hamilton paths (circuits) and Euler paths (circuits). Hamilton graphs are defined in terms of vertices and Euler graphs are defined in terms of edges.

Unfortunately, unlike the condition for an Euler circuit, there is no nice condition to identify when a graph is a Hamilton circuit. It is just a matter of trial and error.

Example 7 Identifying a Hamilton circuit

List a Hamilton circuit for the graph shown.



Solution

A Hamilton circuit is $C \rightarrow B \rightarrow A \rightarrow E \rightarrow F \rightarrow G \rightarrow H \rightarrow D \rightarrow C$.

Not every graph that has a Hamilton circuit has an Euler circuit, and also not every graph that has an Euler circuit has a Hamilton circuit. The graph in Example 7 has a Hamilton circuit but not an Euler circuit. The graph in Figure 23.19 has an Euler circuit but not a Hamilton circuit.

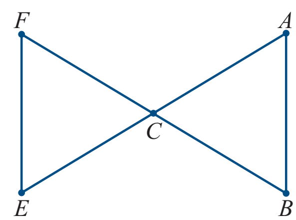


Figure 23.19

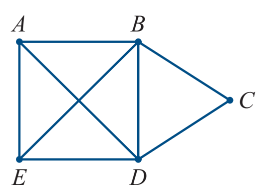
Exercise 23D

Skillsheet TEACHER

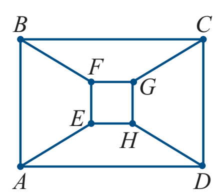
1 i Identify whether each graph below has an Euler circuit, or an Euler path but not an Euler circuit, or neither an Euler circuit nor an Euler path.

ii Name the Euler circuits or paths found.

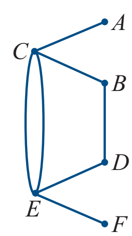
a



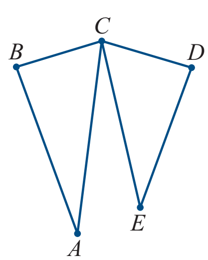
b



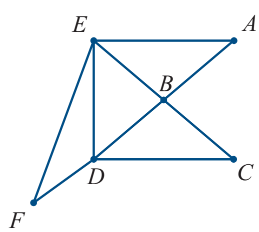
c



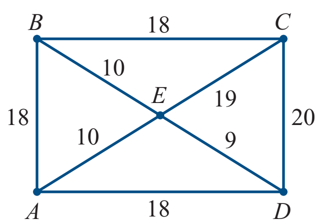
d



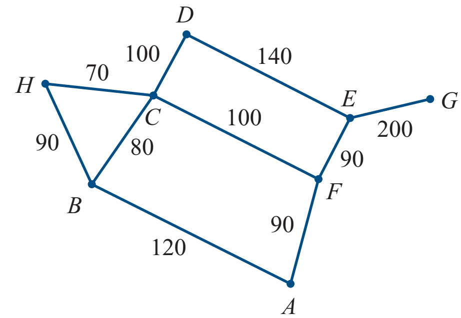
e



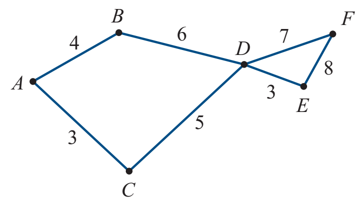
c



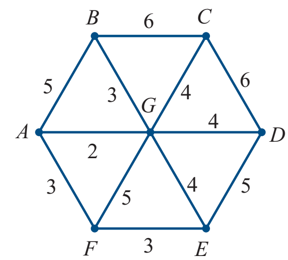
d



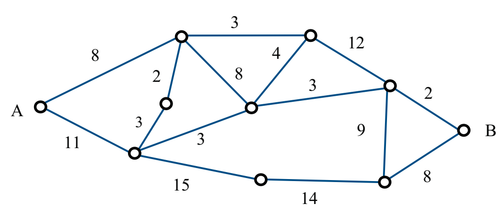
2 By trial and error, find the shortest path from A to E.



3 Find the shortest Hamilton path for the following graph, starting at A.



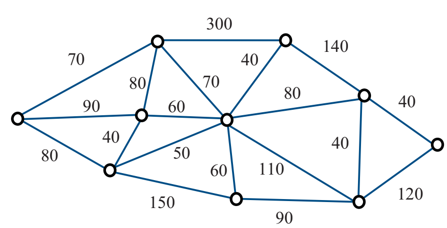
4 In the network opposite, the vertices represent small towns and the edges represent roads. The numbers on each edge indicate the distances (in km) between towns.



a Determine the length of the shortest path between the towns labelled A and B.

b Find the minimal spanning tree for this network and determine its length.

5 In the network opposite, the vertices represent water tanks on a large property and the edges represent pipes used to move water between these tanks. The numbers on each edge indicate the lengths of pipes (in m) connecting different tanks.



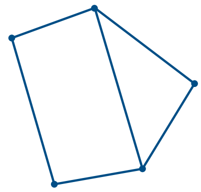
Determine the shortest length of pipe needed to connect all water storages.

MC TEST

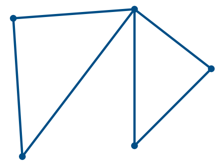
D&D TEST

5 Which of the following graphs does not have an Euler circuit?

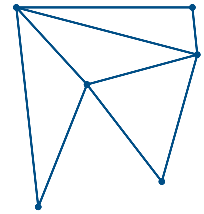
A



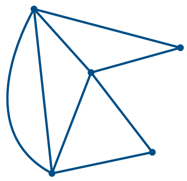
B



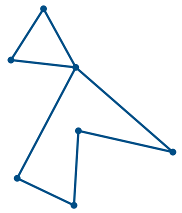
C



D



E



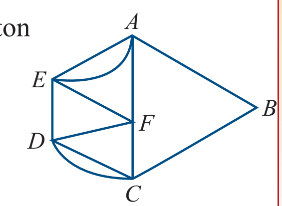
6 A connected planar graph divides the plane into a number of regions. If the graph has eight vertices and these are linked by 13 edges, then the number of regions is:

A 5 **B** 6 **C** 7 **D** 8 **E** 10

7 For the graph shown, which of the following paths is a Hamilton circuit?

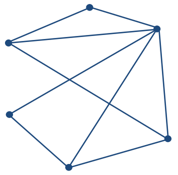
A *ABCDCFDEFAEA* **B** *AEFDCBA* **C** *AFCDEABA*

D *ABCDEA* **E** *AEDCBAF*



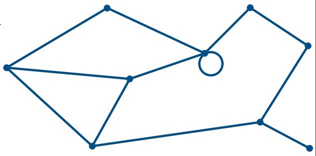
8 The graph opposite has:

A 4 faces **B** 5 faces **C** 6 faces **D** 7 faces **E** 8 faces



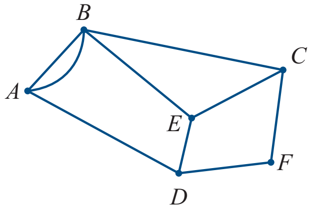
9 The sum of the degrees of the vertices on the graph shown here is:

A 20 **B** 21 **C** 22 **D** 23 **E** 24



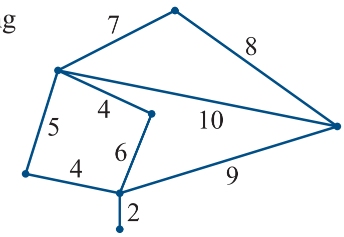
10 For the graph shown, which additional arc could be added to the network so that the graph formed would contain an Euler path?

A *AF* **B** *DE* **C** *AB* **D** *CF* **E** *BF*



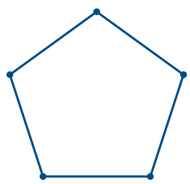
11 For the graph shown here, the minimum length spanning tree has length:

A 30 **B** 31 **C** 33 **D** 34 **E** 26

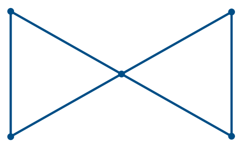


12 Of the following graphs, which one has both Euler and Hamilton circuits?

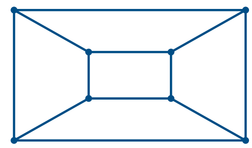
A



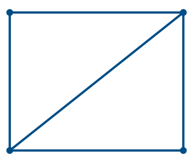
B



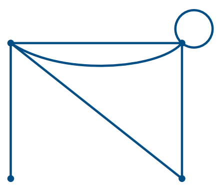
C



D



E



13 A *complete* graph with six vertices is drawn. This network would best represent:

A the journey of a paper boy who delivers to six homes covering the minimum distance

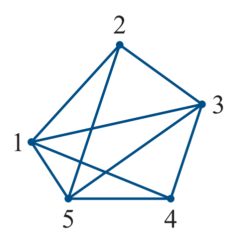
B the cables required to connect six houses to a pay television service that minimises the length of cables needed

C a six-team basketball competition where all teams play each other once

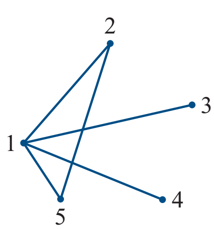
D a project where six tasks must be performed between the start and finish

E the allocation of different assignments to a group of six students [VCAA 2006]

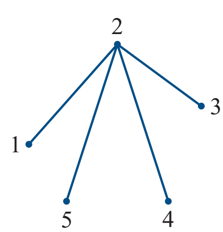
14 Which one of the following is a spanning tree for the graph shown here?



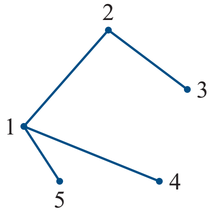
A



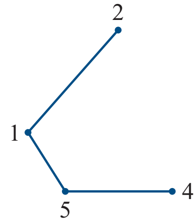
B



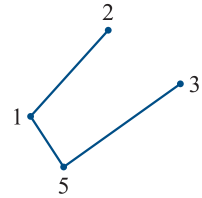
C



D

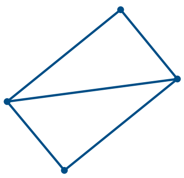


E

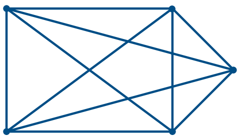


15 Which *one* of the following graphs has an Euler circuit?

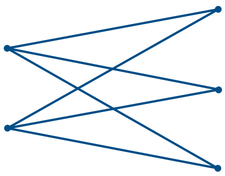
A



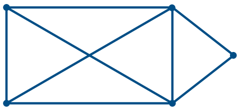
B



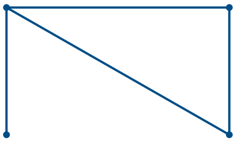
C



D

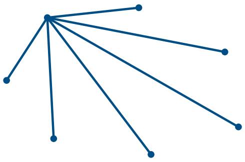


E

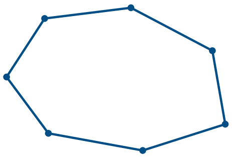


16 Which one of the following graphs provides a counter-example to the statement: 'For a graph with seven vertices, if the degree of each vertex is greater than 2 then the graph contains a Hamilton circuit'?

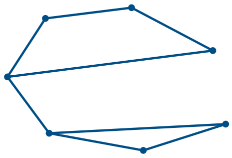
A



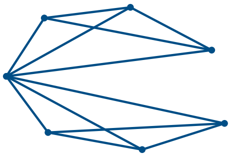
B



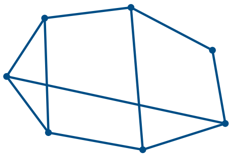
C



D



E



17 A planar graph has 5 vertices and 4 faces. The number of edges is:

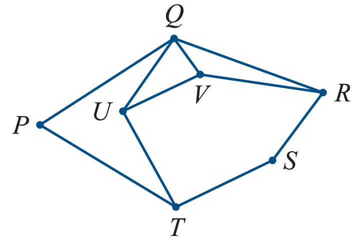
A 5 **B** 6 **C** 7 **D** 8 **E** 9

18 The smallest number of edges for a graph with 10 vertices to be connected is:

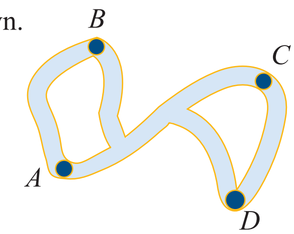
A 6 **B** 7 **C** 8 **D** 9 **E** 10

19 Which one of the following paths is a Hamilton circuit for the graph shown here?

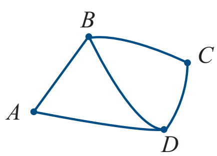
A *PQRSTP* **B** *PQRSTUVP*
C *PQUVRSTP* **D** *PQRSTUVUTP*
E *PQRSTUVRVQUTP*



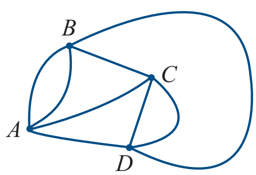
20 Four towns, *A*, *B*, *C* and *D*, are linked by roads as shown. Which of the following graphs could be used to represent the network of roads? Each edge represents a route between two towns



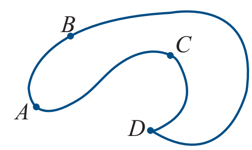
A



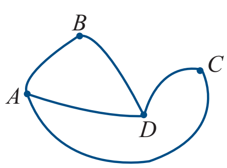
B



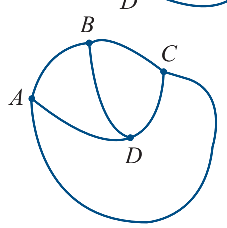
C



D



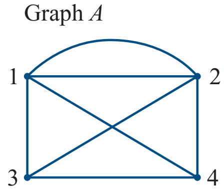
E



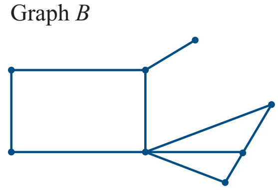
Extended-response questions

1 This question is about the vertices of a graph and the degree of a vertex. In Graph *A* below, there are four vertices (the dots).

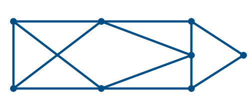
Graph *A*



Graph *B*



Graph *C*



a Complete the table for Graph *B*.

Degree	0	1	2	3	4	5	6	7
Number of vertices								

b Study Graphs *A*, *B* and *C* and then consider the statement:

'In any graph the total number of vertices of odd degree is an even number.'

Is this statement true for Graphs *A*, *B* and *C*? How many vertices of odd degree does each graph have?

iii Fill in the missing entries for the matrix shown for the completed graph formed above.

$$\begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 0 & 1 & 0 & 1 & 1 & 1 \\ B & 1 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 1 & 0 \\ D & 1 & 0 & - & - & - & - \\ E & 1 & 0 & - & - & - & - \\ F & 1 & 0 & - & - & - & - \end{array}$$

b A walker follows the route $A–B–A–F–E–D–C–E–F–A$.

i How far does this person walk?

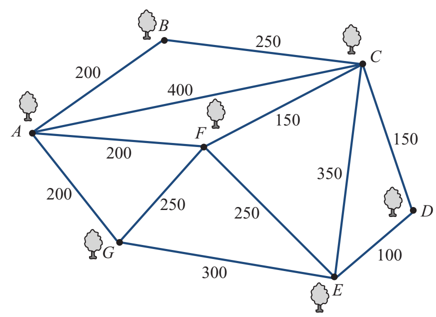
ii Why is the route *not* a Hamilton circuit?

iii Write down a route that a walker could follow that is a Hamilton circuit.

iv Find the distance walked in following this Hamilton circuit.

c It is impossible to start at A and return to A by going along each path exactly once. An extra path joining two campsites can be constructed so that this is possible. Which two campsites need to be joined by a path to make this possible?

4 An estate has large open parklands that contain seven large trees. The trees are denoted as vertices A to G on the network diagram shown. Walking paths link the trees as shown. The numbers on the edges represent the lengths of the paths in metres.



a Determine the sum of the degrees of the vertices of this network.

b One day Jamie decides to go for a walk that will take him along each of the paths between the trees. He wishes to walk the minimum possible distance.

i State a vertex at which Jamie could begin his walk.

ii Determine the total distance, in metres, that Jamie will walk.

Michelle is currently at F. She wishes to follow a route that can be described as the shortest Hamiltonian circuit.

c Write down a route that Michelle can take. [VCAA 2007]

3 Three volunteer workers, Joe, Meg and Ali, are available to help with three jobs. The time (in minutes) in which each worker is able to complete each task is given in the table opposite.
Which allocation of workers to jobs will enable the jobs to be completed in the minimum time?

	Job		
	A	B	C
Joe	20	20	36
Meg	16	20	44
Ali	26	26	44

4 A company has four machine operators and four different machines that they can operate. The table shows the hourly cost in dollars of running each machine for each operator. How should the machinists be allocated to the machines to maximise the hourly output from each of the machines with the staff available?

	Machine			
Operator	*W*	*X*	*Y*	*Z*
A	38	35	26	54
B	32	29	32	26
C	44	26	23	35
D	20	26	32	29

5 A football association is scheduling football games to be played by three teams (the Champs, the Stars and the Wests) on a public holiday. On this day, one team must play at their Home ground, one will play Away and one will play at a Neutral ground.
The costs (in \$000s) for each team to play at each of the grounds are given in the table below.
Determine a schedule that will minimise the total cost of playing the three games and determine this cost.

Team	*Home*	*Away*	*Neutral*
Champs	10	9	8
Stars	7	4	5
Wests	8	7	6

Note: There are two different ways of scheduling the games to achieve the same minimum cost. Identify both of these.

6 A roadside vehicle assistance organisation has four service vehicles located in four different places. The table below shows the distance (in km) of each of these service vehicles from four motorists in need of roadside assistance.

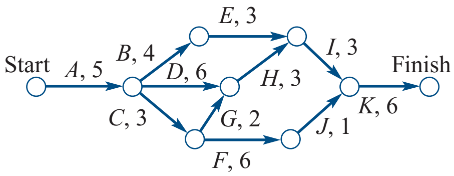
	Motorist			
Service vehicle	*Jess*	*Mark*	*Raj*	*Karla*
A	18	15	15	16
B	7	17	11	13
C	25	19	18	21
D	9	22	19	23

Determine a service vehicle assignment that will ensure that the total distance travelled by the service vehicles is minimised. Determine this distance.
Note: There are two ways that the service vehicles can be assigned to minimise the total distance travelled. Identify both of these.

MC TEST

D&D TEST

11 This graph represents the activity schedule for a project where the component times in days are shown. The critical path for the network of this project is given by:



A *A–B–E–I–K* **B** *A–D–H–I–K* **C** *A–C–G–H–I–K*
D *A–C–F–J–K* **E** *A–D–G–F–J–K*

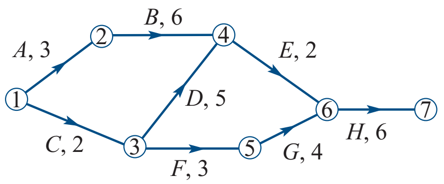
12 The table opposite lists the six activities in a project and the earliest start time, in hours, and the predecessor(s) of each task. The time taken for activity *E* is two hours. Without affecting the time taken for the entire project, the time taken for activity *C* could be increased by:

Task	*Predecessor*	*EST*
A	–	0
B	–	0
C	*A*	8
D	*B*	15
E	*C*	22
F	*D, E*	35

A 0 hours **B** 8 hours **C** 9 hours **D** 11 hours **E** 27 hours

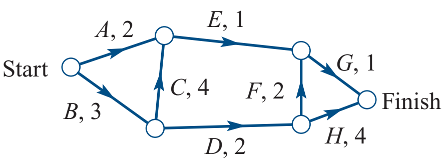
[VCAA pre 2006]

13 The edges in this directed graph correspond to the tasks involved in the preparation of an examination. The numbers indicate the time, in weeks, needed for each task. The total number of weeks needed for the preparation of the examination is:



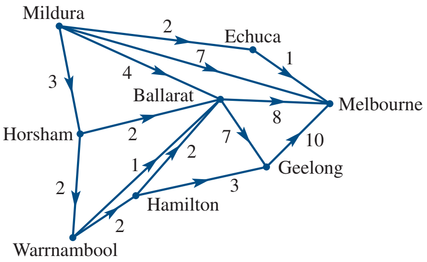
A 14 **B** 15 **C** 16 **D** 17 **E** 27

14 The directed graph represents a manufacturing process with activities and their duration (in hours) listed on the arcs of the graph. The earliest time (in hours) after the start that activity *G* can begin is:

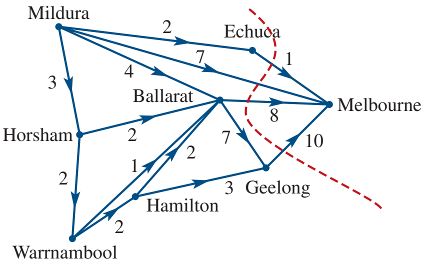


A 3 **B** 5 **C** 6 **D** 7 **E** 8

4 WestAir Company flies routes in western Victoria. The network shows the layout of connecting flight paths for WestAir, which originate in Mildura and terminate either in Melbourne or on the way to Melbourne. On this network the available spaces for passengers flying out of various locations on one morning are listed.



a The network is cut as shown. What does this cut tell us about the maximum number of passengers who could depart Mildura and arrive in Melbourne on this morning using WestAir?



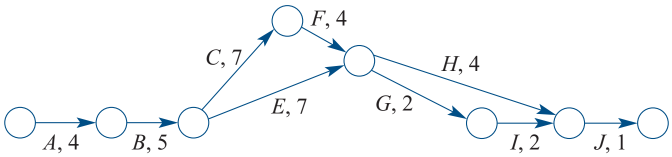
b What is the maximum number of passengers who could travel from Mildura to Melbourne for the morning via WestAir services?

[VCAA pre 2006]

5 LiteAero Company designs and makes light aircraft for the civil aviation industry. They identify 10 activities required for production of their new model, the MarchFly. These, and the associated activity durations, are given in the table opposite.

Activity	*Duration (weeks)*	*Immediate predecessor(s)*
A	4	–
B	5	A
C	7	B
D	6	B
E	7	B
F	4	C
G	2	E, F
H	4	F
I	2	D, G
J	1	H, I

a An incomplete network for this project is shown. Complete the network by drawing and labelling activity D.



(*cont'd.*)

b Use the information from the table to complete the missing earliest and latest start times.

c State the critical path(s) for this network.

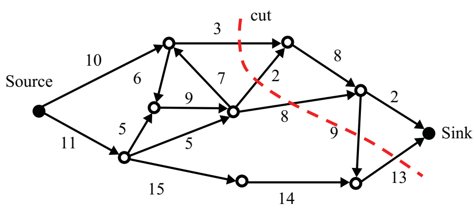
Activity	*Earliest start time*	*Latest start time*
A	0	0
B	4	4
C	9	9
D	9	
E	9	13
F	16	16
G		20
H	20	20
I	22	22
J	24	24

[VCAA pre 2006]

6 A school swimming team wants to select a 4 × 200 metre relay team. The fastest times of its four best swimmers in each of the strokes are shown in the table below. Which swimmer should swim which stroke to give the team the best chance of winning, and what would be their time to swim the relay?

	Stroke			
	Backstroke	*Breaststroke*	*Butterfly*	*Freestyle*
Rob	76	78	70	62
Joel	74	80	66	62
Henk	72	76	68	58
Sav	78	80	66	60

7 In the network opposite, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow in the network. Also shown is a cut that separates the source from the sink.

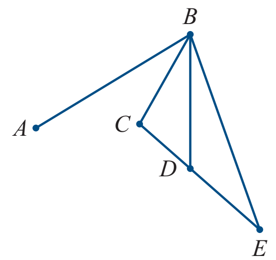


a Determine the capacity of the cut shown.

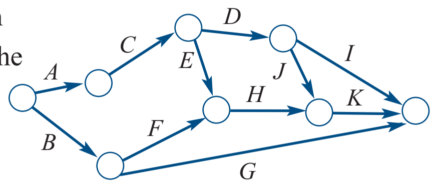
b Determine the maximum flow.

19 What additional arc could be added to the graph to ensure that the resulting graph would contain an Euler circuit?

A AB **B** AC **C** AD **D** AE **E** BC



20 This network represents a project development with activities listed on the arcs of the graph. Which of the following statements must be true?

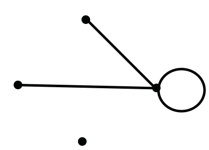


A A must be completed before B can start.
B A must be completed before F can start.
C E and F must start at the same time.
D E and F must finish at the same time.
E E cannot start until A is finished.

21 A connected graph with 12 edges divides a plane into 4 regions. The number of vertices in this graph will be:

A 6 **B** 10 **C** 12 **D** 13 **E** 14

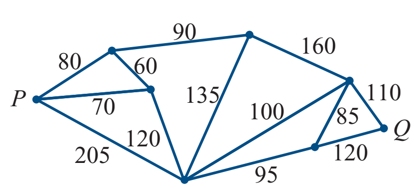
22 An adjacency matrix for the graph opposite could be:



A $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

23 A vehicle is travelling from town P to town Q. The journey requires the vehicle to travel along a network linking suitable fuel stops. The cost, in dollars, of travel between these is shown on the network below, where the nodes represent fuel stops.



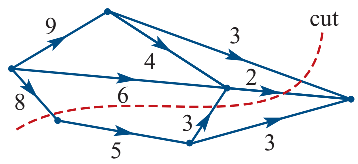
What is the minimum cost, in dollars, for the trip?

A 400 **B** 405 **C** 410 **D** 420 **E** 440

Revision

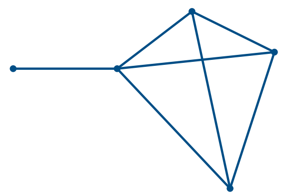
24 The capacity of the cut in the network flow diagram shown is:

A 0 **B** 2 **C** 10 **D** 13 **E** 16



25 The sum of the degrees of the vertices on the graph shown is:

A 12 **B** 13 **C** 14 **D** 15 **E** 16

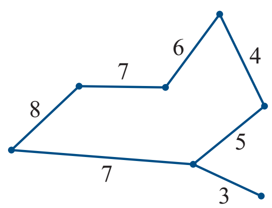


26 A connected planar graph divides the plane into a number of regions. If the graph has 9 vertices and these are linked by 20 edges, then the number of regions is:

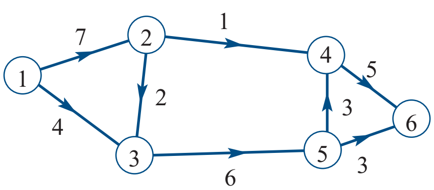
A 11 **B** 13 **C** 21 **D** 27 **E** 31

27 The sum of the weights of the minimum spanning tree of the weighted graph is:

A 2 **B** 30 **C** 32 **D** 33 **E** 35



The following graph relates to questions 28 and 29



28 The maximum flow in the network linking node 1 to node 6 is:

A 5 **B** 6 **C** 7 **D** 8 **E** 9

29 The number of ways that node 6 can be reached from node 1 is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

30 An undirected connected graph has five vertices. Three of these vertices are of even degree and two of these vertices are of odd degree. One extra edge is added. It joins two of the existing vertices.

In the resulting graph, it is *not* possible to have five vertices that are:

A all of even degree
B all of equal degree
C one of even degree and four of odd degree
D three of even degree and two of odd degree
E four of even degree and one of odd degree [VCAA 2009]

31 Five soccer teams played each other once in a tournament. In each game there was a winner and a loser. A table of one-step and two-step dominances summarises the results.

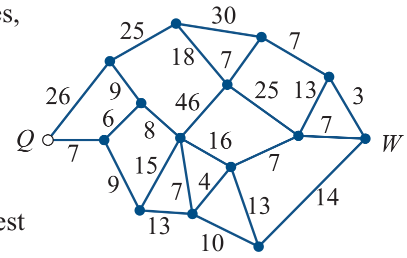
Team	*One-step dominances*	*Two-step dominances*
Aardvarks	1	2
Bears	3	5
Chimps	2	4
Donkeys	3	4
Elephants	1	1

One result in the tournament that must have occurred is that:

A Elephants defeated Bears **B** Elephants defeated Aardvarks
C Aardvarks defeated Donkeys **D** Donkeys defeated Bears
E Bears defeated Chimps [VCAA 2009]

25.2 Extended-response questions

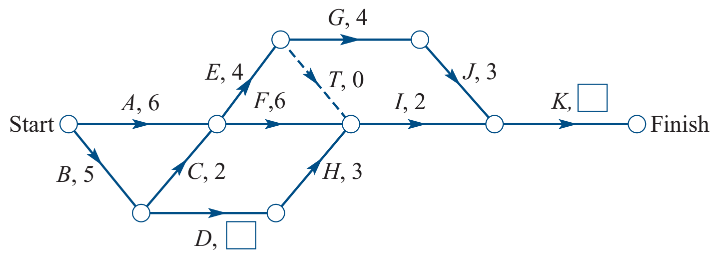
1 The network diagram shows the distances, in kilometres, along a series of roads that connect a quarry, Q, with worksites, shown as nodes.



a One of these worksites is labelled as W.
  i Indicate the shortest path from the quarry to W.
  ii Determine the length, in kilometres, of the shortest path between the quarry Q and the worksite W.

b The engineer at the quarry wants to visit all worksites in the network. Beginning at Q, he wants to pass through each worksite only once before returning to the quarry.
  i What mathematical term describes the route the engineer wants to take?
  ii Show a complete route that the engineer could take to visit each worksite only once before returning to the quarry. [VCAA pre 2006]

2 All the activities and their durations (in hours) in a project at the quarry are shown in the network diagram below. The least time required for completing this entire project is 30 hours.



For each activity in this project, the table on the next page shows the completion time, the earliest starting time and the latest starting time.

(*cont'd.*)

Activity	*Completion time (hours)*	*Earliest starting time (hours)*	*Latest starting time (hours)*
A	6	0	
B	5	0	0
C	2	5	5
D		5	9
E	4	7	7
F	6	7	
G	4	11	11
H	3	9	13
I	2	13	16
J	3	15	15
K		18	18

a Complete the missing times in the table.

b Write down the critical path for this project.

3 A development project involves completing a number of activities as shown in the table. With each activity, there is the optimistic assessment of completion time (i.e. the shortest time likely to occur). Time is measured in days.

Node–link	*Activity*	*Optimistic time (days)*	*Predecessor(s)*
1–2	A	4	–
1–3	B	2	–
2–4	C	1	A
3–4	D	6	B
3–5	E	5	B
3–6	F	7	B
4–7	G	5	C, D
5–7	H	1	E
6–8	I	2	F
7–9	J	10	G, H
8–9	K	6	I

The set of activities can be represented on a directed graph.

a Construct a graph for this project, labelling the activities on the arcs (edges) with their associated shortest durations.

b Determine the earliest start time for each activity from your graph.

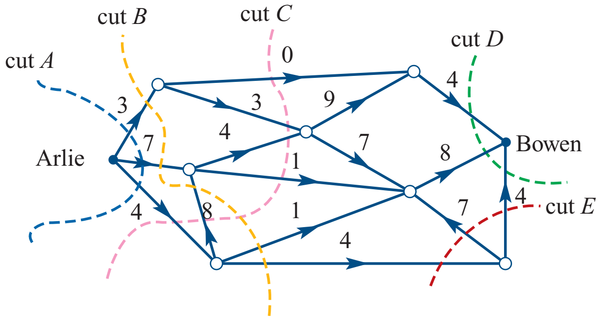
c How long is the estimated project time under this set of activity durations?

d Determine the latest start time for each activity from your graph.

e State the critical path.

f If the final activity, K, had to be delayed, how many days could this delay take before the project schedule was disrupted? [VCAA pre 2006]

4 A train journey consists of a connected sequence of stages formed by edges on the directed network from Arlie to Bowen. The number of available seats for each stage is indicated beside the corresponding edge, as shown in the diagram.



The five cuts, A, B, C, D and E, shown on the network, are attempts to find the maximum number of available seats that can be booked for a journey from Arlie to Bowen.

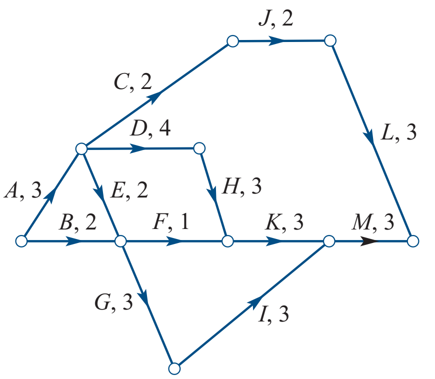
a Write down the capacity of cut A, cut B and cut C.

b Cut E is not a valid cut when trying to find the minimum cut between Arlie and Bowen. Why?

c Determine the maximum number of available seats for a train journey from Arlie to Bowen.

5 The Bowen Yard Buster team specialises in backyard improvement projects. The team has identified the activities required for a backyard improvement. The network diagram on the next page shows the activities identified and the actual times, in hours, needed to complete each activity, that is, the duration of each activity.
The table lists the activities, their immediate predecessor(s) and the earliest starting times (EST), in hours, of each of the activities. Activity X is not yet drawn on the network diagram.

(cont'd.)



	Immediate predecessor(s)	*EST*
A	–	0
B	–	0
C	*A*	3
D	*A*	3
E		3
F	*B*, *E*	5
G	*B*, *E*	5
H	*D*	7
I	*G*	
J	*C*, *X*	8
K	*F*, *H*	10
L	*J*	10
M	*I*, *K*	
X	*D*	7

a Use the information in the network diagram to complete the table.

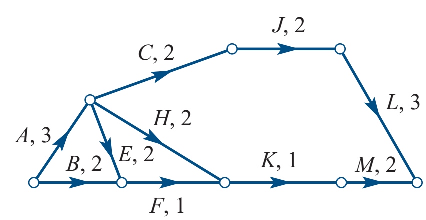
b Draw and label activity *X* on the network diagram above, including its direction and duration.

c The path *A–D–H–K–M* is the only critical path in this project.

i Write down the duration of path *A–D–H–K–M*.

ii Explain the importance of the critical path in completing the project.

6 To save money, Bowen Yard Busters decide to revise the project and leave out activities *D*, *G*, *I* and *X*. This results in a reduction in the time needed to complete activities *H*, *K* and *M* as shown.

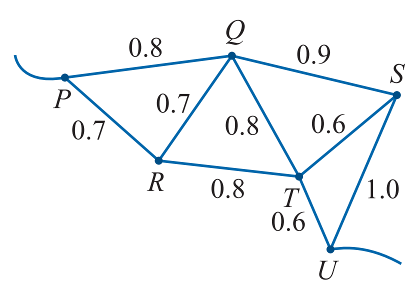


a For this revised project network, what is the earliest starting time for activity *K*?

b Write down the critical path for this revised project network.

c Without affecting the earliest completion time for this entire revised project, what is the latest starting time for activity *M*?

7 A rural town, built on hills, contains a set of roads represented by arcs in the network shown here. The numbers on the network refer to distances along the roads (in kilometres) and the letters refer to intersections of the roads. The arcs without endpoints refer to the two roads in and out of town.



a **i** What is the length of the shortest route through the town from *P* to *U*?

ii A safety officer who enters the town at *P* needs to examine all intersections in the town before leaving from *U* to travel to the next town. To save time, she wants to pass through each intersection only once. State a path through the network of roads that would enable her to do this.

b A technician from the electricity company is checking the overhead cables along each street. The technician elects to follow an Euler path through the network streets (ignoring the roads in and out of town) starting at R and finishing at S.

i Complete the following Euler path: $R–Q–P–R–\square–\square–\square–T–U–S$

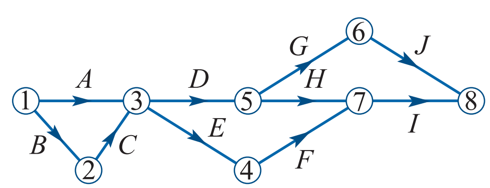
ii How would the technician benefit from choosing an Euler path?

8 The local council plans to turn the main street of the town into a mall. The planning phase involves a number of activities whose normal completion times are supplied in Table 1. Also included in the table are the 'crash time' (possible time to which the activity time can be shortened) and the daily cost of this 'crashing'.

Table 1 Project completion times and costs

Activity	*Normal completion time (days)*	*Crash time (days)*	*Cost of crashing per day* ($)
A	10	8	400
B	5	5	–
C	3	2	400
D	5	4	600
E	4	4	–
F	6	5	500
G	6	4	200
H	7	5	300
I	5	5	–
J	4	3	400

The network for this project is as shown.



a Using normal completion times as given in Table 1, determine the times missing from Table 2.

Table 2 Normal times for job starting

Activity	*Earliest start time (day)*	*Latest start time (day)*
A	0	0
B	0	2
C	5	7
D	10	10
E	10	12
F		16
G	15	
H	15	15
I	22	22
J	21	23

Revision

b **i** State the critical path in this network.

ii Determine the length of the critical path.

c **i** Complete Table 3, taking into account that some of the activities can be crashed, as shown in Table 1, to reduce the total completion time of the project.

ii Determine the shortest time in which the project can now be finished.

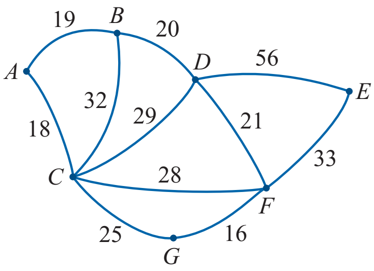
iii Apart from A, what three other activities must be shortened so the project is completed in minimum time?

iv What is the cost of achieving this time reduction for the whole project?

Table 3 Reduced times for job starting using crash data

Activity	*Earliest start time (days)*	*Latest start time (days)*
A	0	0
B	0	1
C	5	6
D	8	8
E	8	8
F	12	12
G	12	15
H	12	12
I	17	17
J		

9 A group of seven towns on an island have been surveyed for transport and communications needs. The towns (labelled A, B, C, D, E, F, G) form the network shown here. The road distances between the towns are marked in kilometres. (All towns may be treated as points being of no size compared to the network lengths.)



a Explain what is meant by the description of the graph as 'planar'.

b The roads between the towns define boundaries used by the local authority to establish rural planning subregions. (That is, the section bounded by roads AB, AC and BC would be one subregion. These subregions are non-overlapping.)
Treating the subregions as faces of the graph (with the exterior of the network as one subregion), the roads as edges and towns as vertices, show that Euler's formula linking the number of vertices, edges and faces in a planar graph, i.e.
number of vertices + number of faces = number of edges + 2, is satisfied.

An inspector of roads is stationed at B. Starting from B, she must travel the complete network of roads to examine them.

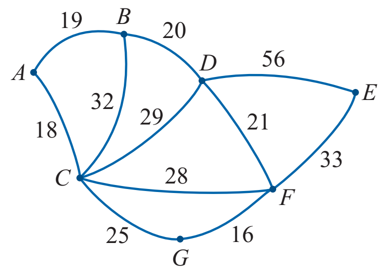
c If she wishes to travel the least distance where will she end up in the network?

d What will that distance be?

e Is the route unique? Briefly justify your answer.

f Determine the shortest distance that a fire truck stationed at E must travel to assist at an emergency at A.

g To establish a cable network for telecommunications on the island, it is proposed to put the cable underground beside the existing roads. What is the minimal length of cable required here if back-up links are not considered necessary; that is, there are no loops in the cable network?

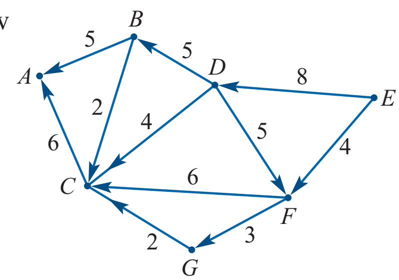


The Island Bank has outlets in each of the towns. The regional assistant manager stationed at C must visit each outlet every second Friday and then return to the office at C.

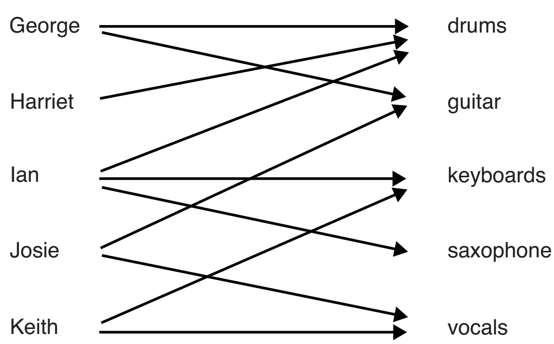
h Treating the towns as vertices and roads as edges in a graph, what is the distance of a journey that forms a Hamilton circuit in the graph?

i What is the length of the trip that gives the optimal (that is, shortest) route to the assistant manager?

j A reservoir at E pumps water through pipes along the network routes shown. The capacities of the flow are given in the digraph shown here in megalitres per day.
Occasionally, there are fire emergencies in the forest beside A and additional flow of water is used. What is the maximum flow that can reach A from E?



10 George, Harriet, Ian, Josie and Keith are a group of five musicians. They plan to form a band where each musician will play one instrument only. The bipartite graph below shows the instrument that each is able to play.



a Given that each musician will play one instrument only:

i which musician must play the guitar?

ii name the instruments that Harriet, Ian and Keith must play

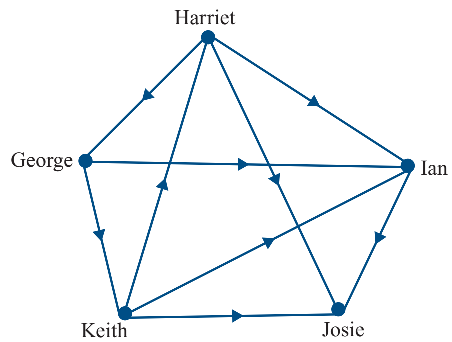
(*cont'd.*)

The five musicians compete in a music trivia game.
Each musician competes once against every other musician.
In each game there is a winner and a loser.
The results are represented in the dominance matrix opposite.

$$\begin{array}{cc} & \text{loser} \\ & \begin{array}{ccccc} G & H & I & J & K \end{array} \\ \text{winner} \begin{array}{c} G \\ H \\ I \\ J \\ K \end{array} & \begin{bmatrix} \mathbf{0} & 0 & 1 & 0 & 1 \\ 1 & \mathbf{0} & 1 & 1 & 0 \\ 0 & 0 & \mathbf{0} & 1 & 0 \\ 1 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 1 & 1 & 1 & \mathbf{0} \end{bmatrix} \end{array}$$

b Explain why the figures in bold in the matrix are all zero.

The incomplete directed graph opposite has been constructed from this matrix. On this graph, draw an arrow from Harriet to George that shows that Harriet won against George.
One of the edges on the directed graph is missing.



c Using the information in the dominance matrix, draw in the missing edge on the graph and show its direction.

The results of each trivia contest (one-step dominances) are summarised in the table opposite.
In order to rank the musicians from first to last in the trivia contest, two-step (two-edge) dominances will be considered.

Musician	*Dominance value (wins)*
George	2
Harriet	3
Ian	1
Josie	1
Keith	3

The incomplete matrix opposite shows two-step dominances.

$$\begin{array}{cc} & \begin{array}{ccccc} G & H & I & J & K \end{array} \\ \begin{array}{c} G \\ H \\ I \\ J \\ K \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & x & 0 \end{bmatrix} \end{array}$$

d Explain the two-step dominance that George has over Ian.

e Determine the value of the entry x in the matrix.

f Taking into consideration both the one-step and two-step dominances, determine which musician was ranked first and which musician was ranked last in the trivia contest.

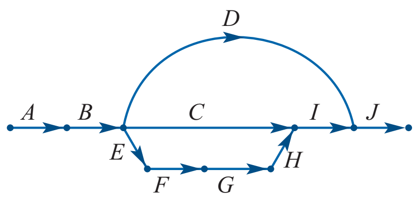
[based on VCAA 2006]

11 Jack's construction company builds a particular type of house using the project plan given in Table 1.

Table 1

Activity	*Description*	*Predecessor*	*Duration (days)*
A	build foundation	–	5
B	build frame	*A*	8
C	build roof	*B*	12
D	do electrical wiring	*B*	5
E	put in windows	*B*	4
F	install insulation	*E*	1
G	install plumbing	*F*	1
H	put on siding	*G*	6
I	paint house	*C*, *H*	3
J	add fixtures/fittings	*D*, *I*	3

A project network for this plan, with activities on arcs, is shown here.



a Using the information in Table 1, determine the times missing from Table 2.

b What is the earliest time in which the project can be completed using the information given in Table 1?

c What is/are the critical path/paths in this network?

d What is the float (slack time) for any activity not on a critical path?

Table 2

Activity	*Earliest start time*	*Latest start time*
A	0	0
B	5	5
C	13	13
D	13	
E		13
F	17	
G	18	18
H	19	19
I	25	25
J	28	28

(*cont'd.*)

Like many construction projects, this plan can have its time reduced by 'crashing' the project, that is, using more resources to finish parts of the job more quickly. Table 3 gives the cost of these reductions and the maximum extent to which each action can be taken.

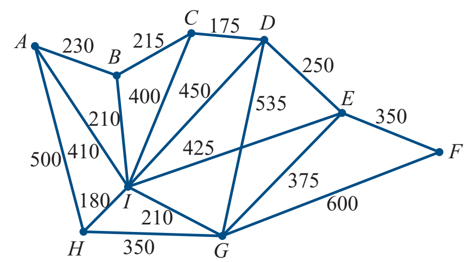
Activity	*Cost per day for reducing activity duration* ($)	*Maximum possible reduction (days)*
A	300	2
B	150	3
C	200	1
D	400	2
E	200	2
H	300	3
I	400	1
J	150	1

Table 3

e Using the information in Table 3, determine the shortest time in which the project can now be completed. Show all working.

f What is the minimum additional cost to achieve this?

12 A company is building the new Bigtown University. The company has constructed nine new faculty buildings in a layout as shown. The minimum distances in metres between adjacent buildings in the university are also shown.

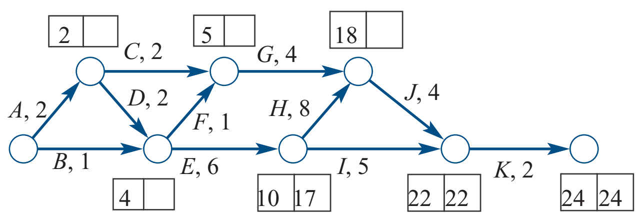


A computer network is to be built to serve the whole university.

a Draw a network that will ensure that all the buildings are connected to the network but that also minimises the amount of cable used. Label each node in the network.

b What is the minimum length of cable required? [VCAA pre 2006]

13 The assembly of machined parts in a manufacturing process can be represented by the following network. The activities are represented by the letters on the arcs and the numbers represent the time taken (in hours) for the activities scheduled.



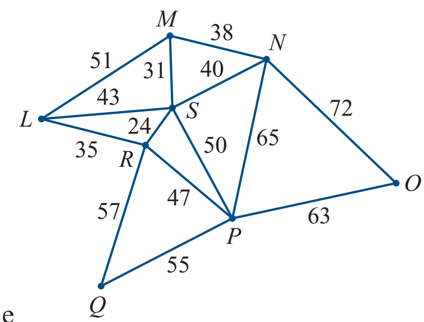
a The earliest start times (EST) for each activity except *G* are given in the table. Complete the table by finding the EST for *G*.

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
EST	0	0	2	2	4	4		10	10	18	22

b What is the shortest time required to assemble the product?

c What is the float (slack time) for Activity *I*?

14 A number of towns need to be linked by pipelines to a natural gas supply. In the network shown, the existing road links between towns L, M, N, O, P, Q and R and to the supply point, S, are shown as edges. The towns and the gas supply are shown as vertices. The distances along roads are given in kilometres.



a What is the shortest distance along roads from the gas supply point S to the town O?

b The gas company decides to run the gas lines along the existing roads. To ensure that all nodes on the network are linked, the company does not need to place pipes along all the roads in the network.

i What is the usual name given to the network within a graph (here, the road system) which links all nodes (towns and supply) and which gives the shortest total length?

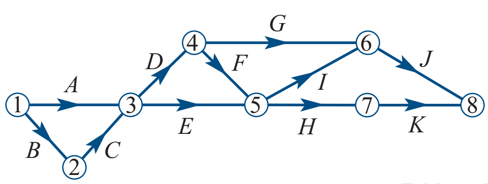
ii Sketch this network.

iii What is the minimum length of gas pipeline the company can use to supply all the towns by running the pipes along the existing roads?

c The gas company decides it wants to run the pipeline directly to any town which is linked by road to its supply at S. Towns not directly connected to S by road will be linked via other towns in the network.

What is the minimum length of pipeline that will enable all towns to be connected to the gas supply under these circumstances?

15 In laying a pipeline, the various jobs involved have been grouped into a set of specific tasks A–K, which are performed in the precedence described in the network below.



a List all the task(s) that must be completed before task E is started.

The durations of the tasks are given in Table 1.

Table 1 Task durations

Task	*Normal completion time (months)*
A	10
B	6
C	3
D	4
E	7
F	4
G	5
H	4
I	5
J	4
K	3

(*cont'd.*)

Revision

b Use the information in Table 1 to complete Table 2.

Table 2 Starting times for tasks

Task	*EST*	*LST*
A	0	0
B	0	
C	6	7
D	10	10
E		11
F	14	14
G	14	18
H	18	20
I	18	
J	23	23
K	22	24

c For this project:

i write down the critical path

ii determine the length of the critical path (that is, the earliest time the project can be completed)

d If the project managers are prepared to pay more for additional labour and machinery, the time taken to complete task *A* can be reduced to 8 months, task *E* can be reduced to 5 months and task *I* can be reduced to 4 months.

Under these circumstances:

i what would be the critical path(s)?

ii how long would it take to complete the project?

16 The pipeline construction team needs tractors at four different worksites. Four tractors are available but these are in four different locations. The cost (in dollars) of providing a tractor at each of the sites from each of the locations is given in Table 3.

Table 3 Cost of providing tractors (in dollars)

	Tractor based at			
Assigned to	*Location 1*	*Location 2*	*Location 3*	*Location 4*
Site 1	1130	830	2010	1140
Site 2	1020	1100	690	850
Site 3	2010	1320	1150	1410
Site 4	960	1210	2100	1530

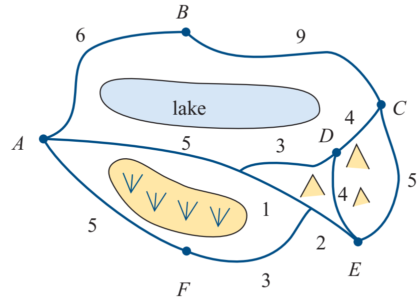
a Use the Hungarian algorithm, or otherwise, to complete the following table. From each location show where the tractors should be sent to minimise the total cost of providing tractors to the pipeline construction team.

Tractor at	*Assign to*
Location 1	Site 4
Location 2	
Location 3	
Location 4	

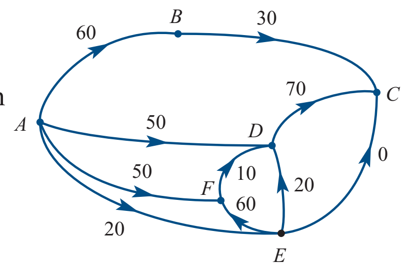
b What is the minimum cost of providing tractors to the pipeline construction team?

[VCAA pre 2006]

17 The map shows six camp sites, A, B, C, D, E and F which are joined by paths. The numbers on the paths show lengths in kilometres of sections of the paths.



The National Park Authority limits the number of people per day who can walk along each of the paths connecting the camp sites as shown in the graph. Note that, due to a landslide, path CE has been blocked and cannot be used.

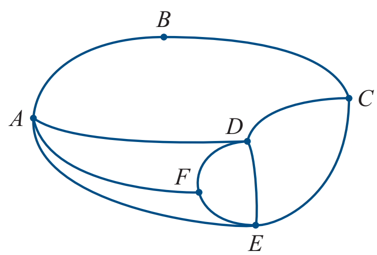


a What is the maximum number of people per day who can travel from A to C using the paths and directions as shown in the graph? Justify your answer.

b The number of people allowed to use the paths each day in the reverse direction is given by the following table.

Path	*Number allowed*
B to A	60
C to B	30
C to D	50
C to E	0
D to E	20
F to E	0
D to F	5
E to A	0
D to A	50
F to A	20

This diagram may be used to assist answering part ***b***.



What is the maximum number of people per day who can travel from C to A?

18 Camp sites A, B, C and D are to be supplied with food. Four local residents, W, X, Y and Z, offer to supply one campsite each. The cost in dollars of supplying one load of food from each resident to each campsite is tabulated.

	W	X	Y	Z
A	30	70	60	20
B	40	30	50	80
C	50	40	60	50
D	60	70	30	70

a Find the two possible matchings between campsites and residents so that the total cost is a minimum.

b State this minimum cost.

Square matrices

As a final example, we could form a matrix we might call M (for males). This matrix contains only the data for the males. As this matrix has four rows and four columns, it is a (4×4) matrix; four rows by four columns. It contains $4 \times 4 = 16$ elements.

$$M = \begin{bmatrix} 173 & 57 & 18 & 86 \\ 179 & 58 & 19 & 82 \\ 195 & 84 & 18 & 71 \\ 184 & 74 & 22 & 78 \end{bmatrix}$$

A matrix like M, with an **equal** number of **rows** and **columns** is called a **square matrix**.

Example 1 Matrix facts

For each of the matrices below, write down its type, order and the number of elements.

Solution

Matrix	Type	Order	No. of elements
$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 0 & 4 \\ 2 & -1 & 6 \end{bmatrix}$	Square matrix no. of rows = no. of columns	(3×3) 3 rows, 3 cols.	9 $3 \times 3 = 9$
$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	Column matrix single column	(3×1) 3 rows, 1 col.	3 $3 \times 1 = 3$
$C = [3 \quad 1 \quad 0 \quad 5 \quad -3 \quad 1]$	Row matrix single row	(1×6) 1 row, 6 cols.	6 $1 \times 6 = 6$

Some notation

In some situations, we would like to talk about a matrix and its elements without having specific numbers in mind. We do this as follows.

For the matrix A, which has n rows and m columns, we write:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,m} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,m} \end{bmatrix}$$

$a_{2,\,3}$

row number column number

Thus:

- $a_{2,1}$ represents the element in the 2nd row and the 1st column
- $a_{1,2}$ represents the element in the 1st row and the 2nd column
- $a_{2,2}$ represents the element in the 2nd row and the 2nd column
- $a_{m,n}$ represents the element in the mth row and the nth column

Note: When there is no confusion, it is common to omit the comma and write $a_{2,\,3}$ as a_{23}.

Example 2 Identifying the elements in a matrix

For the matrices A and B, opposite, write down the values of:

a $a_{1,2}$ **b** $a_{2,1}$ **c** a_{33} **d** b_{31}

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Solution

a $a_{1,2}$ is the element in the 1st row and the 2nd column of A $\quad a_{1,2} = 5$

b $a_{2,1}$ is the element in the 2nd row and the 1st column of A $\quad a_{2,1} = -1$

c a_{33} is the element in the 3rd row and the 3rd column of A $\quad a_{33} = 6$

d b_{31} is the element in the 3rd row and the 1st column of B $\quad b_{31} = 1$

Entering a matrix into a graphics calculator

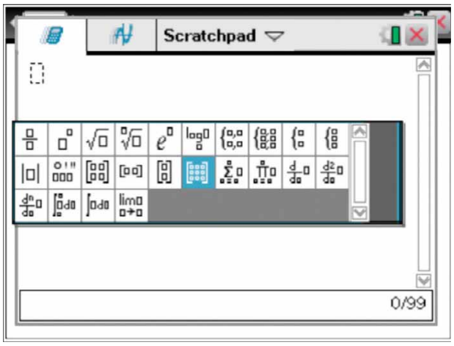
Later in this chapter, you will learn about matrix arithmetic: how to add, subtract and multiply matrices. While it is possible to carry out these tasks by hand, for all but the smallest matrices this is extremely tedious. Most matrix arithmetic is better done with the help of a graphics calculator. However, before you can perform matrix arithmetic, you will need to know how to enter a matrix into your calculator.

How to enter a matrix on the TI-Nspire CAS

Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$.

Steps

1. Press (⌂ on) (or (off/on) then (⌂) on the Clickpad) then (A) to open the **Scratchpad:Calculate.**
   Note: You can also use (⌂ on)**>Documents> New Document>Add Calculator** if preferred.
2. Press (templates) ((ctrl) + (templates) on the Clickpad) to select the **Math Templates**. Use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press (enter).



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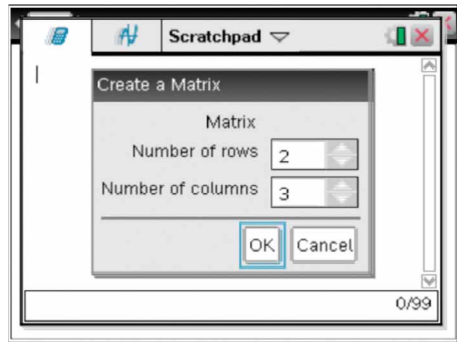
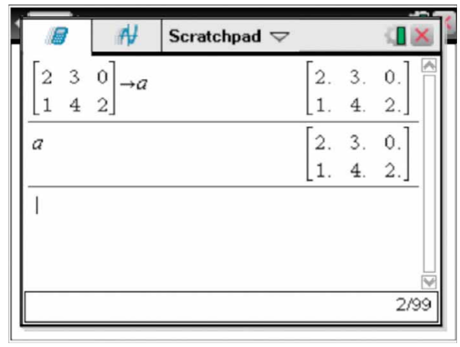
3 Use the ▼ arrow to select the **Number of rows** required. (In this case, the number of rows is 2.) Press (tab) to move to the next entry and repeat for the **Number of columns**. (In this case, the number of columns is 3.) Use (tab) to highlight **OK** and press (enter).



4 Type the values into the matrix template. Use (tab) to move to the required position in the matrix to enter each value.
When the matrix has been completed, press (tab) to move outside the matrix press (ctrl) + (var), followed by [A]. This will store the matrix as the variable a. Press (enter).



5 When you type $\boldsymbol{A}$ (or $\boldsymbol{a}$) it will paste in the matrix $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$. Press (enter) to display.

How to enter a matrix using the ClassPad

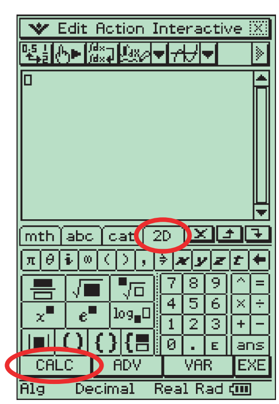
Enter the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$.

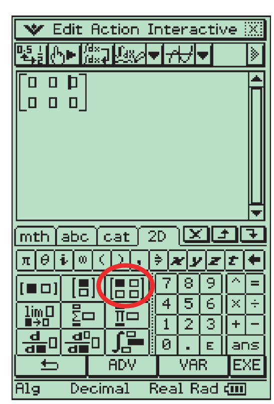
Steps

1 a Locate and open the **Main** (Main) application. Press (Keyboard) to display the hidden keyboard.

b On the keyboard, tap the [2D] tab, followed by the [CALC] menu item at the bottom of the keyboard.

2 Tap the 2 ×2 matrix icon, followed by the 1 × 2 matrix icon. This will add a third column and create a 2 ×3 matrix.





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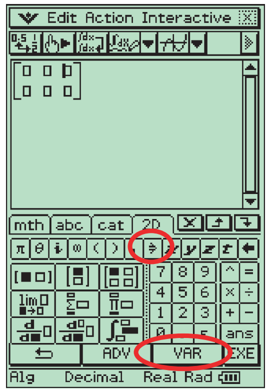
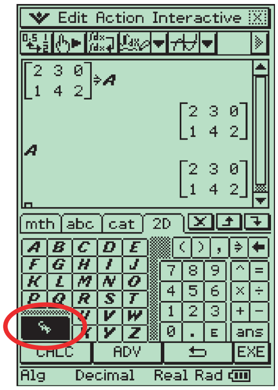
3 Type the values into the matrix template.
Note: Tap at each new position to enter the new value.

4 To assign the matrix the variable name ***A***, move the cursor to the very right-hand side of the matrix, then tap the variable assignment key ⇒ followed by VAR CAP ***A***.
Press EXE to confirm your choice.
Note: Until it is reassigned, ***A*** will represent the matrix as defined above.





Exercise 26A

1 Complete the sentences below that relate to the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 6 \\ -1 & 0 \\ 1 & 3 \\ 4 & -4 \end{bmatrix} \quad E = \begin{bmatrix} 4 & 3 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

a The square matrices are $\square$ and $\square$.

b Matrix B has $\square$ rows.

c The row matrix is $\square$.

d The column matrix is $\square$.

e Matrix D has $\square$ rows and $\square$ columns.

f The order of matrix E is $\square \times \square$.

g The order of matrix A is $\square \times \square$.

h The order of matrix B is $\square \times \square$.

i The order of matrix D is $\square \times \square$.

j There are $\square$ elements in matrix E.

k There are $\square$ elements in matrix A.

l $a_{14} = \square$ m $b_{31} = \square$ n $c_{11} = \square$ o $d_{41} = \square$ p $e_{22} = \square$

q $d_{32} = \square$ r $b_{11} = \square$ s $c_{12} = \square$ t $a_{12} = \square$ u $e_{13} = \square$

2 Enter the following matrices into your calculator and display.

a $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$ b $C = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$ c $E = [1 \quad -1 \quad 2]$ d $F = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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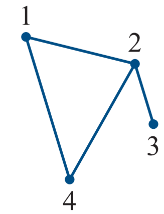
Cambridge University Press

Example 5 Representing a network diagram by a matrix

Represent the network diagram shown opposite by a 4×4 matrix A, where the:

- matrix element $= 1$ if the two points are joined by a line.
- matrix element $= 0$ if the two points are not connected.

Note: Elements are the numbers in the matrix.



Solution

1 Draw in a blank 4×4 matrix, labelling the rows and columns 1, 2, 3, 4 to indicate the points.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

2 Fill in the elements of the matrix row by row, starting at the left-hand top corner:

- $a_{11} = 0$ (there is no line joining point 1 to itself)
- $a_{12} = 1$ (there is a line joining points 1 and 2)
- $a_{13} = 0$ (there is no line joining points 1 and 3)
- $a_{14} = 1$ (there is a line joining points 1 and 4)

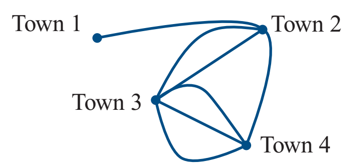
and so on until the matrix is complete.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Note: If a network contains no 'loops' (lines joining points to themselves) the elements in the leading diagonal will always be zero. Knowing this can save a lot of work.

Example 6 Interpreting a matrix representing a network diagram

The diagram opposite shows the roads interconnecting four towns: Town 1, Town 2, Town 3, and Town 4. This diagram has been represented by a 4×4 matrix, A. The elements show the number of roads between each pair of towns.



$$A = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

a In the matrix A, $a_{24} = 1$. What does this tell us?
b In the matrix A, $a_{34} = 3$. What does this tell us?
c In the matrix A, $a_{41} = 0$. What does this tell us?
d What is the sum of the elements in Row 3 of the matrix and what does this tell us?
e What is the sum of all the elements of the matrix and what does this tell us?

Solution

a There is one road between Town 2 and Town 4.
b There are three roads between Town 4 and Town 3.
c There is no road between Town 4 and Town 1.

d 5: The total number of roads between Town 3 and the other towns in the network.

e 14: The total number of different ways you can travel between towns.

Note: For each road, there are two ways you can travel; for example, from Town 1 to Town 2 ($a_{12} = 1$) and from Town 2 to Town 1 ($a_{21} = 1$).

Exercise 26B

1 The table of data opposite gives the number of residents, TVs and computers in three households. Use the table to:

	Residents	*TVs*	*Computers*
Household A	4	2	1
Household B	6	2	3
Household C	2	1	0

a construct a matrix to display the numerical information in the table. What is its order?

b construct a row matrix to display the numerical information in the table relating to Household *B*. What is its order?

c construct a column matrix to display the numerical information in the table relating to computers. What is its order? What does the sum of its elements tell you?

2 The table of data opposite gives the yearly car sales for two car dealers. Use the table to:

Car sales	*Small*	*Medium*	*Large*
Honest Joe's	24	32	11
Super Deals	32	34	9

a construct a matrix to display the numerical information in the table. What is its order?

b construct a row matrix to display the numerical information in the table relating to Honest Joe's. What is its order?

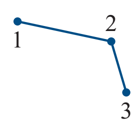
c construct a column matrix to display the numerical information in the table relating to small cars. What is its order? What does the sum of its elements tell you?

3 Convert the 16-digit credit card number 3452 8279 0020 3069 into a 2×8 matrix. List the digits in pairs, one under the other. Ignore spaces.

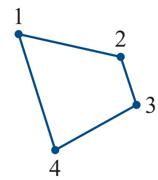
4 Represent each of the following network diagrams by a matrix A using the rules:

- matrix element = 1 if points are joined by a line
- matrix element = 0 if points are not joined by a line.

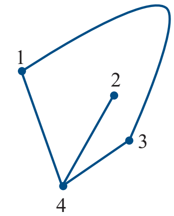
a



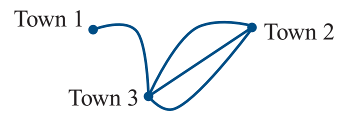
b



c

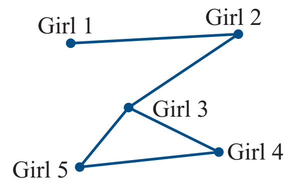


5 The diagram opposite shows the roads interconnecting three towns: Town 1, Town 2 and Town 3.
Represent this diagram by a 3×3 matrix where the elements represent the number of roads between each pair of towns.



6 The network diagram opposite shows a friendship network between five girls: Girls 1 to 5.
This network has been represented by a 5×5 matrix, F, using the rule:

- element = 1 if the pair of girls are friends
- element = 0 if the pair of girls are not friends.



$$F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

a In the matrix F, $f_{34} = 1$. What does this tell us?

b In the matrix F, $f_{25} = 0$. What does this tell us?

c What is the sum of the elements in Row 3 of the matrix and what does this tell us?

d Which girl has the least friends? The most friends?

26.3 Matrix arithmetic: addition, subtraction and scalar multiplication

Equality of two matrices

Two matrices are equal if they are of the same order and each corresponding element is identical in value. It is not sufficient for the two matrices to contain an identical set of numbers; they must also be in the same positions.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ because corresponding elements are equal

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is **not** equal to $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ same numbers but different positions

Matrix addition and subtraction

Adding and subtracting matrices

If two matrices are of the same order (same number of rows and columns), they can be added (subtracted) by adding (subtracting) their corresponding elements.

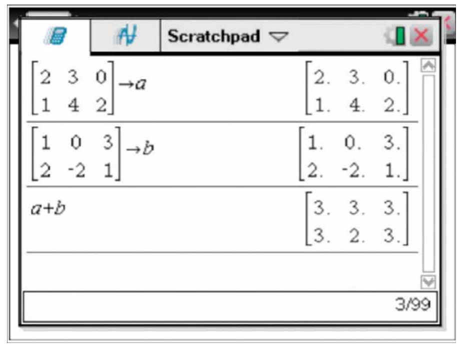
How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$, find

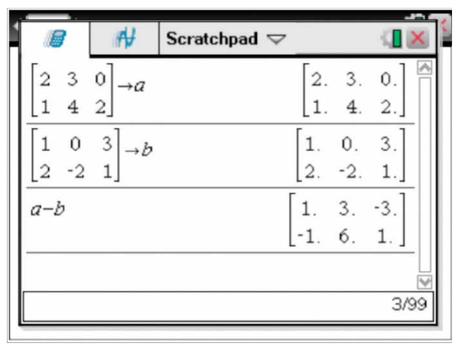
a $A + B$ **b** $A - B$ **c** $3A - 2B$

Steps

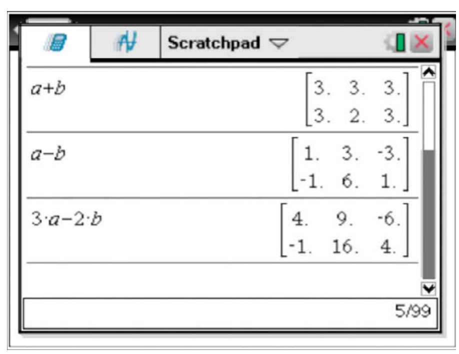
1. Go to **Scratchpad:Calculate**.
   Note: You can also use (on)>**Documents> New Document>Add Calculator** if preferred.



2. Enter the matrices A and B into your calculator.
   Note: Refer to page 695 if you are unsure of how to enter a matrix into your calculator.
   a To determine $A + B$, type $\boldsymbol{a} + \boldsymbol{b}$. Press (enter) to evaluate.
   b To determine. $A - B$, type $\boldsymbol{a} - \boldsymbol{b}$. Press (enter) to evaluate.



   c To determine $3A - 2B$, type $\boldsymbol{3a} - \boldsymbol{2b}$. Press (enter) to evaluate.



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Solution

1 Write down the matrix product and, above each matrix, write down its order. Use this information to determine whether the matrix product is defined and its order.

$$AB = \underset{(2 \times 2)}{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \underset{(2 \times 1)}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

AB is defined because the number of columns in A equals the number of rows in B.
The order of AB is (2×1).

2 To determine the matrix product:

- **i** multiply each element in the row matrix by the corresponding element in the column matrix
- **ii** add the results
- **iii** write down your answer

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 0 \times 3 \\ 2 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$

In principle, if you can multiply a row matrix by a column matrix, you can work out the product between any two matrices, provided it is defined. However, because you have to do it for every possible row/column combination, it soon gets beyond the most patient and careful human being. For that reason, in practice we make use of technology to do it for us.

Using a calculator to multiply two matrices

We will illustrate how to use a calculator to multiply matrices by evaluating the matrix product in the football score example given earlier.

How to multiply two matrices using the TI-Nspire CAS

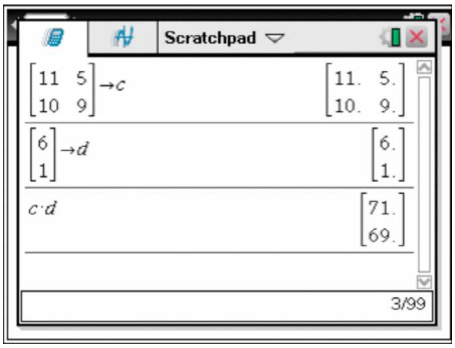
If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

Steps

1 Go to **Scratchpad:Calculate.**
Note: You can also use (on)**>Documents> New Document>Add Calculator** if preferred.

2 Enter the matrices C and D into your calculator.

3 To calculate matrix CD, type $\boldsymbol{c} \times \boldsymbol{d}$.
Press (enter) to evaluate.
Note: You must put a multiplication sign between the c and d.



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Key ideas and chapter summary

Matrix (plural: matrices)

A **matrix** is a rectangular array of numbers or symbols (elements) enclosed in brackets.

$[5]$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

are all examples of matrices.

Row matrix (row vector)

A **row matrix** contains a **single row** of elements.

$\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ is an example of a row matrix.

Column matrix (column vector)

A **column matrix** contains a **single column** of elements.

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an example of a column matrix.

Square matrix

A **square** matrix has an **equal number of rows and columns**.

$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is an example of a square matrix.

The null (zero) matrix

A **null** (**zero**) matrix contains only zeros.

$[0]$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all examples of null matrices.

Order of a matrix

The **order** of a matrix is defined by the number of rows and columns.

A matrix with m rows and n columns is said to be of order $m \times n$ (read 'm by n').

For example:

$\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ is a (1×3) matrix: one row and three columns

$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is a (2×2) matrix: two rows and two columns

The order of a matrix is important in determining whether it can be added to, subtracted from or multiplied by another matrix.

Locating an element in a matrix

The location of each element in the matrix is specified by its row and column number.

For example, in the matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$ the element

- $a_{12} = -1$ is in the 1st row and 2nd column
- $a_{21} = 4$ is in the 2nd row and 1st column

For a $(m \times n)$ matrix, the **number of elements** $= \boldsymbol{m \times n}$.

For example:

$\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ is a (1×3) matrix and has $1 \times 3 = 3$ elements.

$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is a (2×2) matrix and has $2 \times 2 = 4$ elements.

12 X is a 3×2 matrix. Y is a 2×3 matrix. Z is a 2×2 matrix. Which of the following matrix expressions is *not* defined?

A XY **B** YX **C** $XZ - 2X$ **D** $YX + 2Z$ **E** $XY - YX$

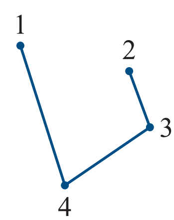
13 $A = [1\ 1\ 1\ 1]$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 4 \end{bmatrix}$.

Which of the following will generate a matrix that displays the mean of 3, 5, 2, 4 ?

A $\frac{1}{4}(A + B)$ **B** $\frac{1}{2}(A + B)$ **C** $\frac{1}{4}B$ **D** $\frac{1}{4}AB$ **E** $\frac{1}{4}BA$

14 The diagram opposite is to be represented by a matrix, A, where:

- element $= 1$ if the two points are joined by a line
- element $= 0$ if the two points are not connected.



The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

15 The matrix equation $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ generates the following pair of simultaneous linear equations:

A $x + 4y = 2$, $3x - 2y = 1$

B $3x - 2y = 1$, $x + 4y = 2$

C $x + 2y = 1$, $4x + y = 2$

D $3x + 2y = 1$, $x - 4y = 2$

E $x + 2y = 3$, $4x - y = -2$

3 Write down the matrix and use the rule

$$\det(C) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d - b \times c.$$

Evaluate.

Use the formula

$$C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to evaluate.

c

$$C = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \det(C) = \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 4 = -2$$

$$\therefore C^{-1} = \frac{1}{\det(C)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(-2)} \begin{bmatrix} 3 & -4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -1.5 & 2 \\ 1 & -1 \end{bmatrix}$$

Using a graphics calculator to determine the determinant and inverse of an $n \times n$ matrix ($n \geq 2$)

There are rules for finding the inverse of a square matrix of any size, but they are extremely complicated and take huge amounts of time to compute by hand. So, in practice, we use a calculator to find the inverse of all but a (2×2) matrix. In fact, you can use a calculator to compute the inverse of a (2×2) matrix if you wish, but it is often just as quick to do it by hand. The same goes for calculating determinants, although the determinant of a (2×2) matrix is computed much more quickly by hand.

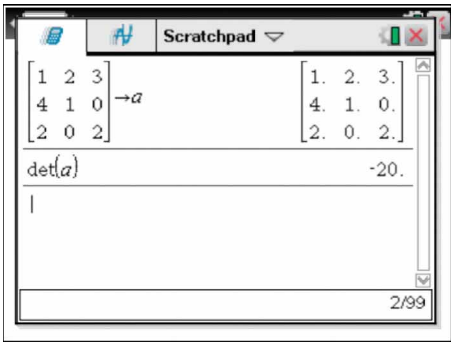
How to find the determinant and inverse of a matrix using the TI-Nspire CAS

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

Steps

1. Go to **Scratchpad: Calculate**
2. Enter the matrix A into your calculator.
3. To calculate $\det(A)$, type **det(*a*)** and press (enter) to evaluate.

   Note: det( ) can also be accessed using (menu)**>Matrix & Vector>Determinant**.



ISBN 978-1-107-65590-4

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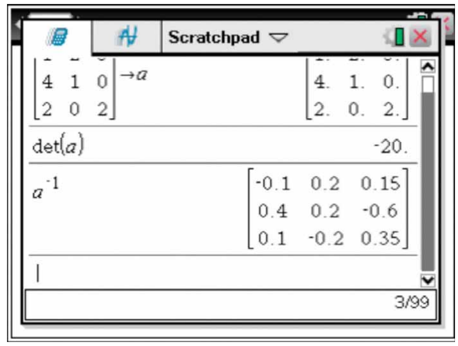
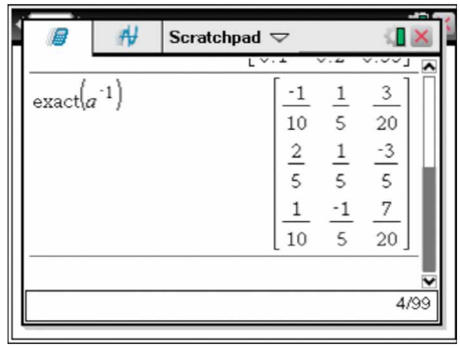
4 To calculate the inverse matrix A^{-1}, type ***a***^ — **1** and press (enter) to evaluate. If you want to see the answer in fractional form, enter as **exact(*a*^ — 1)** and press (enter) to evaluate.



Notes:

1 When the elements in the matrix to be inverted are whole numbers, the elements of the inverse will always be whole numbers or fractions. If this is the case, and the inverse you obtain with your calculator contains decimals, it is worth converting it to fractional form. Although the mode is set to **Approximate** (or decimal), we can make the display show fractions using **exact(. . .)** in front of the command. **exact(. . .)** can also be pasted from the **Catalog** (📖).

2 If the matrix has no inverse, the calculator will respond with the error message **Singular matrix**.



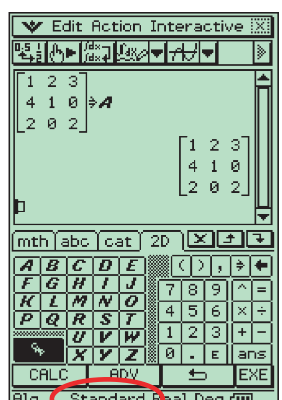
How to find the determinant and inverse of a matrix using the ClassPad

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

Steps

1 Enter the matrix A into your calculator.

Note: Change the status of the calculator to **Standard** in order for fractions to be displayed. Tapping on **Decimal** will change the calculator to **Standard**.



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Solution

1 Rewrite the equations in matrix form.

$$\begin{bmatrix} 3 & 4.5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Let $A = \begin{bmatrix} 3 & 4.5 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $C = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$.

3 Provided that $\det(A) \neq 0$, the solution in matrix form is $X = A^{-1}C$.

In this case:

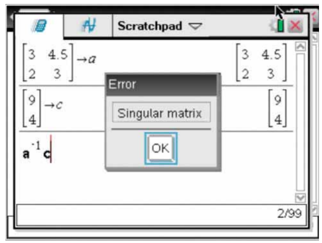
$$\det(A) = \begin{vmatrix} 3 & 4.5 \\ 2 & 3 \end{vmatrix} = 0,$$

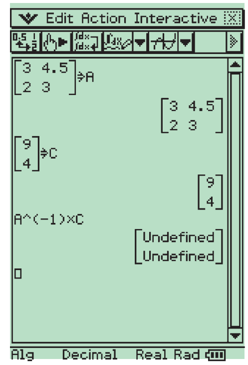
but let us see what happens.

Then $AX = C$ or $X = A^{-1}C$ $(\det(A) \neq 0)$

4 Enter the matrices A and C into your calculator.

5 Attempt to solve the matrix equation by evaluating the matrix product $A^{-1}C$.





6 The calculator gives an error message: **Singular matrix** or **Undefined**. This is because $\det(A) = 0$. The system of equations does not have a unique solution. Write down your conclusion.

No unique solution as $\det(A) = 0$.

The power of the matrix method for solving systems of linear equations becomes apparent when we solve a system of three or more equations.

Example 7 Solving a set of three simultaneous linear equations using the inverse matrix

Solve using matrix methods:

$$\begin{aligned} 3x + 4y - 2z &= -5 \\ 2x + 3y &= -1 \\ x + 2y + 3z &= 3 \end{aligned}$$

Solution

1 Rewrite the equations in matrix form.

$$\begin{bmatrix} 3 & 4 & -2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

2 By identifying the matrices A, X and C, rewrite the matrix equation in the form $AX = C$.

Let $A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $C = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$

Note: There is no z term in the second equation, so its coefficient is zero.

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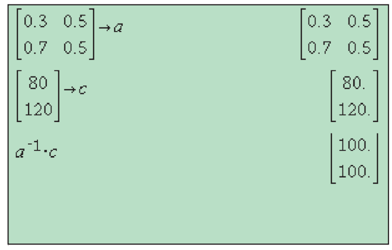
4 Enter the matrices A and C into your calculator and solve the matrix equation by evaluating the matrix product $A^{-1}C$.



5 Write down your answer.

Therefore, $X = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ or $x = 100$ and $y = 100$.

Exercise 27B

1 Write each of the following systems of linear equations in matrix form.

a $\begin{aligned} 3x + 2y &= 2 \\ 2x + 5y &= 4 \end{aligned}$

b $\begin{aligned} 3x + 5y &= 6 \\ 2x + 4y &= 3 \end{aligned}$

c $\begin{aligned} x + 2y &= 1 \\ 2x - 3y &= 2 \end{aligned}$

d $\begin{aligned} x - 3y &= 7 \\ -2x + y &= 4 \end{aligned}$

e $\begin{aligned} -3x - 2y &= 2 \\ x + 2y &= -1 \end{aligned}$

f $\begin{aligned} 3x + 4y - 2z &= 5 \\ 2x + 3y + 5z &= 2 \\ x + 2y + 3z &= 3 \end{aligned}$

g $\begin{aligned} 5x \qquad - 2z &= 3 \\ x - y + z &= 2 \\ x + y + z &= 1 \end{aligned}$

h $\begin{aligned} x + y - 2z + w &= 3 \\ 2x - y + z - w &= 2 \\ x + 2y + z + w &= 1 \\ 2x - 3y + 2z - 2w &= 0 \end{aligned}$

2 Give two explanations of how a system of two linear equations can have no unique solution.

3 What is the condition for the matrix equation $AX = C$ **not** to have a unique solution?

4 By evaluating an appropriate determinant, determine which of the following matrix equations have no solution.

a $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

c $\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

e $\begin{bmatrix} 2.5 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

f $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

g $\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

h $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$

i $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

j $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

k $\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Exercise 27D

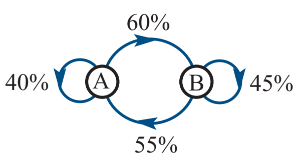
Setting up a transition matrix

1 Complete the following transition matrices:

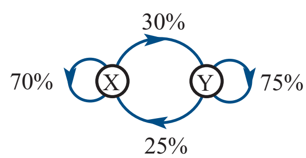
a $\begin{bmatrix} 0.75 & 0.05 \\ 0.25 & \end{bmatrix}$ **b** $\begin{bmatrix} 0.90 & 0.15 \\ & 0.85 \end{bmatrix}$ **c** $\begin{bmatrix} 0.80 & \\ 0.20 & 0.65 \end{bmatrix}$ **d** $\begin{bmatrix} 0.50 & 0.33 \\ & \end{bmatrix}$

2 The diagrams below describe a series of transitions between the states indicated. Construct a transition matrix that can be used to represent each of these diagrams. Use columns to define the starting points. Convert the percentages to proportions.

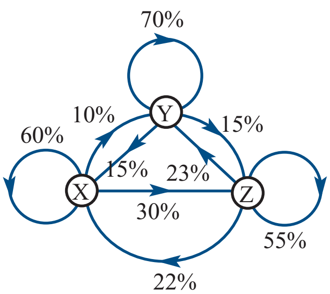
a



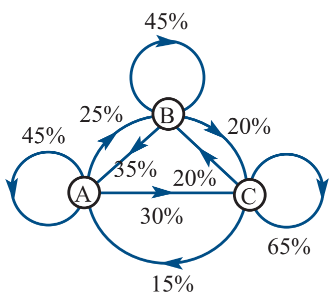
b



c



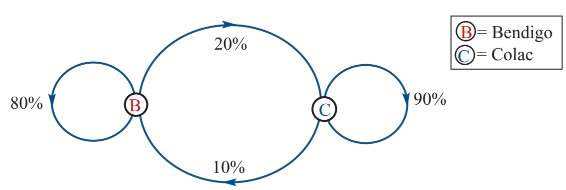
d



Interpreting transition matrices

Let us return to the car rental problem at the start of this section. As we saw then, the following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns at Bendigo and Colac.

$$T = \begin{array}{c c} \text{rented in} & \\ \begin{array}{cc} B & C \end{array} & \\ \begin{bmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{bmatrix} & \begin{array}{l} B \\ C \end{array} \textit{ returned to} \end{array}$$



Using this information alone, a number of predictions can be made.

For example, if 50 cars are rented in Bendigo this week, the transition matrix predicts that:

- 80% or 40 of these cars will be returned to Bendigo next week ($0.80 \times 50 = 40$)
- 20% or 10 of these cars will be returned to Colac next week ($0.20 \times 50 = 10$).

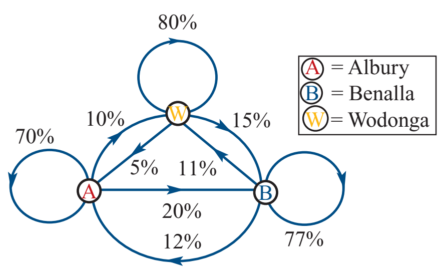
Further, if 40 cars are rented in Colac this week, the transition matrix predicts that:

- 10% or 4 of these cars will be returned to Bendigo next week ($0.10 \times 40 = 4$)
- 90% or 36 of these cars will be returned to Colac next week ($0.90 \times 40 = 36$).

Example 13 Interpreting a transition matrix

The following transition matrix, T, and its transition diagram can be used to describe the weekly pattern of rental car returns in three locations: Albury, Wodonga and Benalla.

$$\begin{array}{cccccl} & \multicolumn{3}{c}{\textit{rented in}} & & \\ & A & W & B & & \\ T = & \multicolumn{3}{c}{\left[\begin{array}{ccc} 0.7 & 0.05 & 0.12 \\ 0.1 & 0.8 & 0.11 \\ 0.2 & 0.15 & 0.77 \end{array}\right]} & \begin{array}{c} A \\ W \\ B \end{array} & \textit{returned to} \end{array}$$



Use the information in transition matrix T and its transition diagram to answer the following questions.

a What percentage of cars rented in Wodonga each week are predicted to be returned to:
 i Albury? **ii** Benalla? **iii** Wodonga?

b Two hundred cars were rented in Albury this week. How many of these cars do we expect to be returned to:
 i Albury? **ii** Benalla? **iii** Wodonga?

c What percentage of cars rented in Benalla each week are *not* expected to be returned to Benalla?

d One hundred and sixty cars were rented in Albury this week. How many of these cars are expected to be returned to either Benalla or Wodonga?

Solution

a i 0.5 or 5% ii 0.15 or 15% iii 0.80 or 80%

b i $0.70 \times 200 = 140$ cars ii $0.20 \times 200 = 40$ cars iii $0.10 \times 200 = 20$ cars

c $11 + 12 = 23\%$ or $100 - 77 = 23\%$

d 20% of 160 + 10% of 160 = 48 cars

Exercise 27E

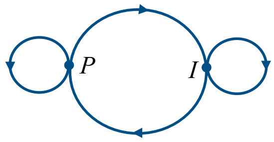
Interpreting transition matrices

1 Each time people attend the movies they buy either a bag of popcorn (P) or an ice cream (I). Experience has shown that:

- 85% of people who buy popcorn this time will buy popcorn next time
- 15% of people who buy popcorn this time will buy an ice cream next time
- 75% of people who buy an ice cream this time will buy an ice cream next time
- 25% of people who buy ice cream this time will buy popcorn next time.

a Construct transition matrix and transition diagram that can be used to describe this situation. Use the models below.

$$
\begin{array}{cc}
 & \text{this time} \\
 & \begin{array}{cc} P & \quad I \end{array} \\
T = & \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array}\right] \begin{array}{l} P \\ I \end{array} \text{next time}
\end{array}
$$



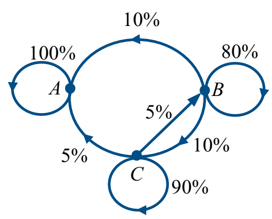
b Eighty people are seen buying popcorn at the movies. How many of these are expected to buy popcorn next time they go to the movies?

c Sixty people are seen buying an ice cream at the movies. How many of these are expected to buy popcorn next time they go to the movies?

d On another occasion, 120 people are seen buying popcorn and 40 are seen buying an ice cream. How many of these people are expected to buy an ice cream next time they attend the movies?

2 On Windy Island, sea birds are observed nesting at three sites, A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of these sea birds between these sites from year to year.

$$
\begin{array}{cc}
 & \text{this year} \\
 & \begin{array}{ccc} A & \quad B & \quad C \end{array} \\
T = & \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \begin{array}{l} A \\ B \text{ next year} \\ C \end{array}
\end{array}
$$



a What percentage of sea birds nesting at site B this year were expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

b This year, 850 sea birds were observed nesting at site B. How many of these birds are expected to:

i still nest at site B next year ? **ii** move to site A to nest next year?

c This year, 1150 sea birds were observed nesting at site A. How many of these birds are expected to nest at:

i site A next year? **ii** site B next year? **iii** site C next year?

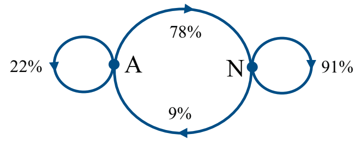
d What does the '1' in column *A, row A* of the transition matrix indicate?

3 A car insurance company finds that:

- 22% of car drivers who are involved in an accident this year (A) are also expected to be involved in an accident next year
- 9% of drivers who are *not* involved in an accident this year (N) are expected to be involved in an accident next year.

The transition diagram that can be used to describe this situation is shown below.

$$\begin{array}{cc} & \text{this year} \\ & \begin{array}{cc} A & N \end{array} \\ T = & \begin{bmatrix} 0.22 & 0.09 \\ 0.78 & 0.91 \end{bmatrix} \begin{array}{l} A \\ N \end{array} \text{next year} \end{array}$$



a In 2011, 84 000 drivers insured with the company were *not* involved in an accident.

  i How many of these 84 000 drivers were expected *not* to be involved in an accident in 2012?

  ii How many of these 84 000 drivers were expected to be involved in an accident in 2012?

b In 2011, 25 000 drivers insured with the company were involved in an accident.

  i How many of these drivers were expected to be involved in an accident in 2012?

  ii How many of these same drivers were expected to be involved in an accident in 2013?

  iii How many of these same drivers were expected to be involved in an accident in 2014?

Using matrix methods to make predictions with a transition matrix

We return again to the car rental problem.

The car rental firm now plans to buy 90 new cars and base 50 in Bendigo and 40 in Colac.

Given this pattern of rental car returns, the questions the manager would like answered are:

- 'If we start off with 50 cars in Bendigo, and 40 cars in Colac, how many cars will be available for rent at Bendigo and Colac after one week, two weeks, etc?'
- 'What will happen in the long term? Will the numbers of cars available for rent each week from each location vary from week to week or will it settle down to some fixed value?'

We saw in the previous section how to predict the number of cars returned to each of the sites after one week. In principle, we could extend this method to answer the remaining questions, but it would be extremely tedious and time-consuming. However, using our knowledge of matrices, we can develop a method that will enable us to answer such questions in a computationally efficient manner.

As before, we start with the transition matrix that describes how the cars move between Bendigo and Colac:

$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

Initially, there are 50 cars in Bendigo, and 40 cars in Colac.

We now construct a column matrix S_0, called an **initial state matrix**, to show this situation.

$$S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

After one week, the number of cars at each branch will change.

Construct a new state matrix S_1 to show the number of cars at each branch after one week.

Then,

$$\begin{aligned} S_1 &= T\,S_0 \\ &= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 50 \\ 40 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 \times 50 + 0.1 \times 40 \\ 0.2 \times 50 + 0.9 \times 40 \end{bmatrix} \\ \text{or } S_1 &= \begin{bmatrix} 44 \\ 46 \end{bmatrix} \end{aligned}$$

Thus, after one week there will be 44 cars in Bendigo and 46 in Colac.

What is the situation after two weeks?

Following the same pattern,

$$S_2 = T\,S_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 44 \\ 46 \end{bmatrix} = \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix}$$

Thus, after two weeks (theoretically) there will be 39.8 cars in Bendigo and 50.2 in Colac.

After three weeks

$$S_3 = T\,S_2 = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} 39.8 \\ 50.2 \end{bmatrix} = \begin{bmatrix} 36.9 \\ 53.1 \end{bmatrix}$$

Thus, after three weeks (theoretically) there will be 36.9 cars in Bendigo and 53.1 in Colac.

A pattern is now emerging. So far we have seen that:

$$S_1 = TS_0, \quad S_2 = TS_1, \quad S_3 = TS_2$$

Continuing this pattern, we can write a general rule that links successive state matrices:

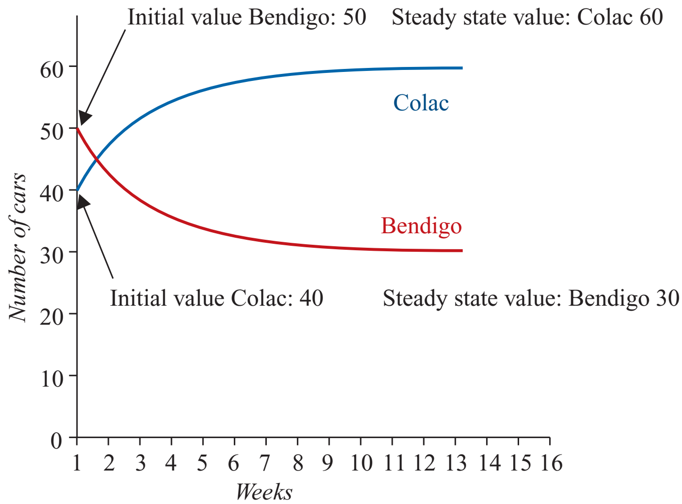
$$S_n = TS_{n-1}$$

Using this rule, we find the following:

$$\begin{array}{cccccccccc} \text{Week } 4 & \text{Week } 5 & \text{Week } 6 & \text{Week } 7 & & \text{Week } 11 & \text{Week } 12 & \text{Week } 13 & \text{Week } 14 & \text{Week } 15 \\ \begin{bmatrix} 34.8 \\ 55.2 \end{bmatrix} & \begin{bmatrix} 33.4 \\ 56.6 \end{bmatrix} & \begin{bmatrix} 32.4 \\ 57.6 \end{bmatrix} & \begin{bmatrix} 31.6 \\ 58.4 \end{bmatrix} & \cdots & \begin{bmatrix} 30.4 \\ 59.6 \end{bmatrix} & \begin{bmatrix} 30.3 \\ 59.7 \end{bmatrix} & \begin{bmatrix} 30.2 \\ 59.8 \end{bmatrix} & \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix} & \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix} \end{array}$$

What you should notice is that as the weeks go by, the number of cars at each of the locations starts to settle down to what we call the **steady state solution** given by the matrix $\begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}$.

The theoretical **steady state solution** is 30.1 (in practice, 30) cars at the Bendigo branch and 59.9 (in practice, 60) cars at the Colac branch and, **it will not change from then on**. This can be seen in the graph (the points have been joined to guide the eye).



Note: In the **steady state**, cars are still moving between Bendigo and Colac, but the number of cars rented in Bendigo and returned to Colac is balanced by the number of cars rented in Colac and returned to Bendigo.

Example 14 Making step-by-step predictions with a transition matrix

Let us return to the factory problem in Example 12. The factory has a large number of machines. The machines can be in one of two states: operating (O) or broken (B). Broken machines are repaired and come back into operation and vice versa. On a given day, the situation is described by the transition matrix. Columns define the machine states at the start of the day.

$$T = \begin{array}{c c} & \begin{array}{cc} O & B \end{array} \\ \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \end{array}$$

At the start of a particular day, 80 machines are operating and 20 are broken. How many machines are in operation and how many are broken after:

a one day? **b** three days?

Solution

1 Write down the transition matrix.

$$T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$$

2 Write down a column matrix with S_0 representing the initial operational state of the machines.

$$S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

3 To determine the operational state of the machines after one day, form the product $S_1 = T\,S_0$ and evaluate.

$$S_1 = T\,S_0 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 69 \\ 31 \end{bmatrix}$$

4 Write down your conclusion.

After one day, 69 machines are operational and 31 are broken.

5 To find the operational state of the machines after three days, we must first find the operating state of the machines after two days. Form the product $S_2 = T S_1$ and evaluate.

$$S_2 = T S_1 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \begin{bmatrix} 69 \\ 31 \end{bmatrix} = \begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix}$$

6 The operating state of the machines after 3 days is then given by the product $S_3 = T S_2$. Evaluate.

$$S_3 = T S_2 = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \begin{bmatrix} 60.2 \\ 39.8 \end{bmatrix} = \begin{bmatrix} 53.16 \\ 46.84 \end{bmatrix}$$

7 Write down your conclusion.

After three days, 53 machines are operating and 47 are broken.

While we can use repeated matrix multiplication to work out successive states of an evolving situation such as the car rental problem, there is a more efficient method when we want to investigate long-term behaviour.

If we follow through the process step by step we have:

$$S_1 = T S_0$$
$$S_2 = T S_1 = T(T S_0) = T^2 S_0$$
$$S_3 = T S_2 = T(T S_1) = T^2 S_1 = T^2(T S_0) = T^3 S_0 \text{ and so on.}$$

Thus we can write for the nth step:

$$S_n = T^n S_0$$

We now have a simple rule for determining the state matrix after n steps.

Example 15 Making predictions with a transition matrix using the rule $S_n = T^n S_0$

Let us return to the factory problem of Examples 12 and 14. On a given day, the situation is described by the transition matrix. Columns define the machine states at the start of a day.

$$T = \begin{array}{c c} & \begin{array}{cc} O & B \end{array} \\ \begin{array}{c} O \\ B \end{array} & \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \end{array}$$

At the start of a particular day, 80 machines are operating and 20 are broken. How many machines are in operation and how many are broken after 10 days?

Solution

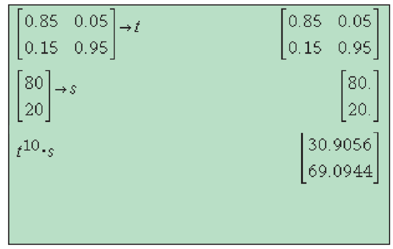
1 Write down the transition matrix, T, and initial state matrix, S_0. Enter the matrices into your calculator. Use T and S.

$$T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix} \quad S_0 = \begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

2 To find out how many machines are in operation and how many are broken after 10 days, write down the rule $S_n = T^n S_0$ and substitute $n = 10$ to give $S_{10} = T^{10} S_0$.

$$S_n = T^n S_0$$
$$\therefore S_{10} = T^{10} S_0$$

3 Enter the expression $T^{10}S$ into your calculator and evaluate.



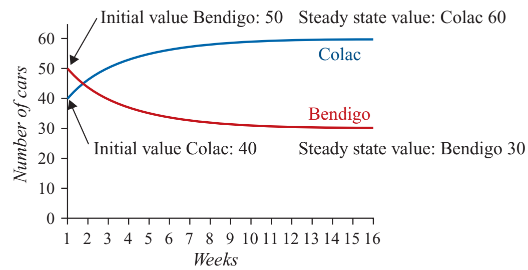
4 Write down your answer in matrix form, then in words.

$$S_{10} = \begin{bmatrix} 30.9 \\ 69.1 \end{bmatrix}$$

After 10 days, 31 machines will be operational and 69 broken.

Finding the steady state solution

In the car rental problem, we found that even though the number of cars returned to each location varied from day to day, eventually it settled down to a steady state solution in which the number of cars at each location remained the same. See the graph opposite.



Although we arrived at this conclusion by repeated calculations, we can arrive at the solution much faster.

Finding the steady state solution using the rule $S_n = T^n S_0$

If S_0 is the initial state matrix, then the **steady state** matrix S, is given by

$$S = T^n S_0$$

as n tends to infinity (∞).

While in practice we cannot evaluate T^n for $n = \infty$, we find that, depending on the circumstances, values of n around 15 to 30 can often give a very close approximation to the steady state solution.

Example 16 **Estimating the steady state solution**

For the car rental problem, the transition matrix is $T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$.

Estimate the steady state solution by calculating S_n for $n = 10$, 17 and 18.

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Solution

1 Write down the transition matrix T and initial state matrix S_0. Enter the matrices into your calculator. Use T and S.

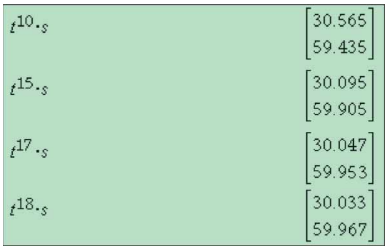
$$T = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \quad S_0 = \begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

2 Use the rule $S_n = T^n S_0$ to write down the expression for the nth state for $n = 10$.

$$S_n = T^n S_0$$
$$\therefore S_{10} = T^{10} S_0$$

3 Enter the expression $T^{10}S$ into your calculator and evaluate.

4 Repeat the process for $n = 15$, 17 and 18.



5 Write down your answer in matrix form, then in words. This result agrees with the graphical result arrived at earlier.

$$S_{15} = \begin{bmatrix} 30.1 \\ 59.9 \end{bmatrix}, \quad S_{17} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}, \quad S_{18} = \begin{bmatrix} 30.0 \\ 60.0 \end{bmatrix}$$

There appears to be a steady state solution with 30 cars at Bendigo and 60 at Colac.

Note: To establish a steady state to a given degree of accuracy, in this case one decimal place, at least two successive state matrices must agree to this degree of accuracy.

Exercise 27F

1 For the transition matrix $T = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$ and an initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$:

a Use the relationship $S_n = TS_{n-1}$ to determine: **i** S_1 **ii** S_2 **iii** S_3

b Determine the value of T^5.

c Use the relationship $S_n = T^n S_0$ to determine **i** S_2 **ii** S_3 **iii** S_7

d Calculate $S_n = T^n S_0$ for $n = 10$, 15, 21 and 22 to show that the steady state solution is close to $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$.

2 For the transition matrix $T = \begin{bmatrix} 0.7 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$ and an initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$:

a Use the relationship $S_n = TS_{n-1}$ to determine: **i** S_1 **ii** S_2 **iii** S_3

b Use the relationship $S_n = T^n S_0$ to determine: **i** S_2 **ii** S_3 **iii** S_7

c Calculate $S_n = T^n S_0$ for $n = 10$, 15, 17 and 18 to show that the steady state solution is close to $\begin{bmatrix} 247.1 \\ 129.4 \\ 223.5 \end{bmatrix}$.

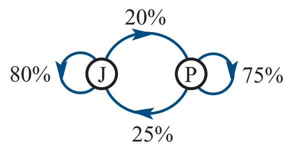
ISBN 978-1-107-65590-4 Cambridge University Press

3 Two fast-food outlets, Jill's and Pete's, are located in a small town. In a given week:

- 80% of people who go to Jill's return the next week
- 20% of people who go to Jill's go to Pete's the next week
- 25% of people who go to Pete's go to Jill's the next week
- 75% of people who go to Pete's return the next week

a Construct a transition matrix T of the form

$$\text{Next week: } \begin{array}{c c} & \text{This week} \\ & \begin{array}{cc} J & P \end{array} \\ \begin{array}{c} J \\ P \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$



to describe this situation.

b In the initial week, 800 people go to the fast-food outlets, 400 to Jill's and 400 to Pete's. Write down a column matrix S_0 that describes this situation.

c How many of the 800 people do we expect to go to Jill's in the next week? How many go to Pete's?

d How many of the 800 people do we expect to go to Jill's after five weeks' time? How many go to Pete's?

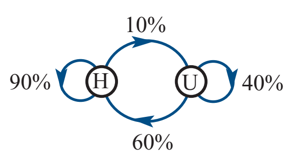
e In the longer term, how many of the 800 people do we expect to go to Jill's? How many go to Pete's?

4 Imagine that we live in a world in which people are either 'happy' or 'unhappy'. However, the way people feel changes from day to day. In the imagined world:

- 90% of people who are happy today will be happy tomorrow
- 10% of people who are happy today will be unhappy tomorrow
- 40% of people who are unhappy today will be happy tomorrow
- 60% of people who are unhappy today will be unhappy tomorrow.

a Construct a transition matrix, T, of the form

$$\text{Tomorrow: } \begin{array}{c c} & \text{Today} \\ & \begin{array}{cc} H & U \end{array} \\ \begin{array}{c} H \\ U \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$



to describe this situation.

b On a given day, out of 2000 people, 1500 are happy and 500 are unhappy. Write down a column matrix, S_0, that describes this situation.

c The next day, how many people do we expect to be 'happy' and how many 'unhappy'?

d After four days, how many people do we expect to be 'happy' and how many 'unhappy'?

e In the long term, how many people do we expect to be 'happy' and how many 'unhappy'?

5 In another model of this world, people can be 'happy', 'neither happy nor sad', or 'sad'. However, the way people feel can change from day to day. The table below shows the ways in which people's feelings may vary from day to day in this imaginary world, and the proportion of people involved.

	Today		
Tomorrow	*Happy*	*Neither*	*Sad*
happy	0.80	0.40	0.35
neither	0.15	0.30	0.40
sad	0.05	0.30	0.25

a Construct a transition matrix, T, to describe this situation. Use the columns to define the situation today and the rows to describe the situation tomorrow.

b On a given day, out of 2000 people, 1200 are 'happy', 600 are 'neither happy nor sad' and 200 are 'sad'. Write down a column matrix, S_0, that describes this situation.

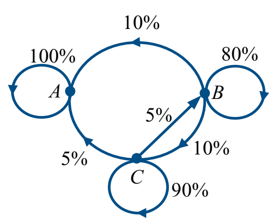
c The next day, how many people do we expect to be happy, neither happy nor sad, or sad?

d After five days, how many people do we expect to be happy, neither happy nor sad, or sad?

e In the long term, how many of the 2000 people do we expect to be happy, neither happy nor sad, or sad?

6 On Windy Island, sea birds are observed nesting at three sites, A, B and C. The following transition matrix and accompanying transition diagram can be used to predict the movement of these sea birds between these sites from year to year.

$$\begin{array}{c} \text{this year} \\ \begin{array}{cc} & \begin{array}{ccc} A & \;\;B & \;\;\;C \end{array} \\ T = & \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix} \begin{array}{l} A \\ B \text{ next year} \\ C \end{array} \end{array} \end{array}$$



In year 1, 10 000 sea birds were observed nesting at each site.

a Write down the state matrix, S_1, that describes this situation.

b Use the rule $S_{n+1} = TS_n$ to:

i determine S_2, the state matrix for year 2

ii predict the number of sea birds nesting at site B in year 3

c Without calculation, write down the number of sea birds predicted to nest at each of the three sites in the long term. Explain why this can be done without calculation.

7 For the transition matrix $T = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ and the state matrix $S_1 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$:

a Use the relationship $S_{n+1} = TS_n$ to determine: **i** S_2 **ii** S_4

b Use the relationship $S_{n+1} = TS_n + R$, where $R = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, to determine: **i** S_2 **ii** S_3

c Use the relationship $S_{n+1} = TS_n - B$, where $B = \begin{bmatrix} -20 \\ 20 \end{bmatrix}$, to determine: **i** S_2 **ii** S_3

8 Returning to Exercise 6:

To help solve the problem of having all the birds eventually nesting at site A, the ranger suggests that 2000 sea birds could be removed from site A each year and relocated in equal numbers to sites B and C.

The state matrix, S_2, is now given by

$S_2 = TS_1 + N$

where $S_1 = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix}$, $T = \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix}$ and $N = \begin{bmatrix} -2000 \\ 1000 \\ 1000 \end{bmatrix}$.

a Evaluate:

i S_2 **ii** S_3 (assuming that $S_3 = TS_2 + N$)

When this plan is tested, it is found that the sea birds tended to be concentrated at site C. The desired outcome is to have equal numbers of birds nesting at each of the sites. An alternative plan is to remove 1500 seabirds from site A each year and relocate all of them at site B.

The state matrix, S_1, is now given by

$S_2 = TS_1 + M$

where $S_1 = \begin{bmatrix} 10000 \\ 10000 \\ 10000 \end{bmatrix}$, $T = \begin{bmatrix} 1.0 & 0.10 & 0.05 \\ 0 & 0.80 & 0.05 \\ 0 & 0.10 & 0.90 \end{bmatrix}$ and $M = \begin{bmatrix} -1500 \\ 1500 \\ 0 \end{bmatrix}$.

b Evaluate:

i S_2 **ii** S_3 (given that $S_3 = TS_2 + M$)

c Without performing any calculations, write down the number of birds expected to nest at each site in the long term.

9 Given $T = \begin{bmatrix} 0.60 & 0.20 \\ 0.40 & 0.80 \end{bmatrix}$, $B = \begin{bmatrix} 0.20 & 0 \\ 0 & 0.10 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, evaluate

a i $S_1 = TS_0 + BS_0$ **ii** $S_2 = TS_1 + BS_1$

b i $S_1 = DS_0$ where $D = T + B = \begin{bmatrix} 0.80 & 0.20 \\ 0.40 & 0.90 \end{bmatrix}$ **ii** $S_2 = D^2 S_0$

10 a Given $R = \begin{bmatrix} 1.5 & -0.20 \\ 0.40 & 0.50 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, evaluate:

i $S_1 = RS_0$ **ii** $S_2 = RS_1$ **iii** $S_2 = R^2 S_0$

b Using $S_n = RS_{n-1}$, evaluate (correct to the nearest whole number):

i $S_3 = R^3 S_0$ **ii** S_6 **iii** S_{20}

11 On Icy Island, there are three penguin rookeries, A, B and C. The following matrix, G, and the accompanying transition diagram can be used to describe the way that the penguins move between the three rookeries each year. The matrix also allows for a 20% increase in penguin numbers at each rookery each year. This increase is due to an excess of births over deaths.

In 2012, there were 4000 penguins at each site.

$$\begin{array}{cc} & \textit{this year} \\ & \begin{array}{ccc} A & B & C \end{array} \\ G = & \begin{bmatrix} 0.90 & 0.10 & 0.05 \\ 0.10 & 1.00 & 0.05 \\ 0.20 & 0.10 & 1.10 \end{bmatrix} \begin{array}{l} A \\ B \ \textit{next year} \\ C \end{array} \end{array}$$

a Write down a state matrix, S_{2012}, that can be used to represent this situation.

b Use the relationship $S_{2013} = GS_{2012}$ to determine S_{2013}.

c Use the relationship $S_{n+1} = GS_n$ to determine (correct to the nearest whole number):

i S_{2014} **ii** S_{2015} **iii** S_{2016} **iv** S_{2017}

d Using the information contained in the state matrices S_{2016} to S_{2017}, describe how penguin numbers at each site are expected to change during the period 2012 to 2016.

12 A farm has a large population of both rabbits (R) and foxes (F).
The initial population matrix

$$P_0 = \begin{bmatrix} 1000 \\ 200 \end{bmatrix} \begin{array}{l} R \\ F \end{array}$$

shows that, initially, there were 1000 rabbits and 200 foxes on the farm.

a In total, how many rabbits and foxes were living on the farm?

The number of rabbits and foxes on the farm one month later can be predicted using the matrix equation $P_1 = NP_0$, where N is the matrix

$$\begin{array}{cc} & \textit{this month} \\ & \begin{array}{cc} R & F \end{array} \\ N = & \begin{bmatrix} 1.1 & -0.20 \\ 0.20 & 0.80 \end{bmatrix} \begin{array}{l} R \\ F \end{array} \ \textit{next month} \end{array}$$

b Determine P_1, the population matrix after one month.

c In total, how many rabbits and foxes are expected to be living on the farm after one month?

Assume that the population matrix for the following months can be determined as follows: $P_2 = NP_1$, $P_3 = NP_2$ or $P_n = GP_{n-1}$

d **i** Calculate P_3, P_6, and P_9.

ii Describe how the number of rabbits and foxes on the farm is expected to change in the first 9 months.

e By what month would we expect the rabbit population on the farm to disappear?

MC TEST D&D TEST

Review

Key ideas and chapter summary

The identity matrix

An **identity** (**unit**) matrix, I, is a square matrix with ones down the leading diagonal and zeros elsewhere.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$ are all examples of identity matrices

The determinant of a matrix

The **determinant** of a matrix, A, is written as $\det(A)$ or $|A|$.

If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then $\det(A) = \begin{bmatrix}a & b\\c & d\end{bmatrix} = a \times d - b \times c$

Only **square** matrices have determinants.

For example, if $A = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}$ then

$$\det(A) = \begin{vmatrix}1 & 3\\3 & 4\end{vmatrix} = 1 \times 4 - 3 \times 3 = -5$$

For higher order matrices, a calculator is used to calculate the determinant.

The inverse of a matrix

The **inverse of a matrix**, A, is written as A^{-1} and has the property that $AA^{-1} = A^{-1}A = I$.

Only **square matrices** have inverses.

The **inverse** of a matrix is **not defined** if $\det(A) = 0$.

Determining the inverse of a matrix

If $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then its inverse, A^{-1}, is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$$

provided $\dfrac{1}{ad - bc} \neq 0$, that is, provided $\det(A) \neq 0$.

For example, if $A = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}$ then

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\-c & a\end{bmatrix} = \frac{1}{-5}\begin{bmatrix}4 & -3\\-3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-0.8 & 0.6\\0.6 & -0.2\end{bmatrix}$$

For higher order matrices, a calculator is used to determine the inverse.

Representing systems of linear equations with matrices

The set of linear equations $\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$ can be written in matrix form as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$.

For example, the set of equations $\begin{aligned} x + 3y &= 1 \\ 3x + 4y &= 2 \end{aligned}$

can written in matrix form as $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A similar pattern follows for sets of three, four, five etc. equations.

Matrix solution of a system of linear equations

Provided that $\det(A) \neq 0$, the set of linear equations defined by $AX = C$ has the solution $X = A^{-1}C$. The order of multiplication is important.

For example, the solution of $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

is given by
$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & -0.2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} \end{aligned}$$

If $\mathbf{det}(\boldsymbol{A}) = \mathbf{0}$, then the system of equations has **no unique solution**. The equations are either **inconsistent** (at least two of the graphs do not cross), or **dependent** (at least two of the graphs are identical).

Power of a matrix

The **power** of a matrix is defined in the same way as the powers of numbers: $A^2 = A \times A$, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$ and so on.

Only **square** matrices can be raised to a power.

$\boldsymbol{A^0}$ is defined to be I, the **identity matrix**.

Transition matrix

Transition matrices are square and have the property that the sum of the columns equals one. For example,

$T = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$ could be a transition matrix.

Initial state matrix

The **initial state matrix**, $\boldsymbol{S_0}$, defines the starting state of a system. For a two-state system, $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ could be an initial state matrix.

Review

Steady state solution matrix	The **steady state matrix**, S, represents the final state of a system. The final state of a system can be estimated by calculating $T^n S_0$ for a large value of n.

Skills check

Having completed this chapter you should be able to:

- recognise an identity matrix
- calculate the determinant of a matrix
- know the properties of an inverse matrix
- find the inverse of a square matrix using a calculator
- use determinants to test a system of linear equations for solutions
- use inverse matrices to solve systems of linear equations
- determine the power of a square matrix using a calculator
- find the steady state solution of a system given the initial state, S_0, and the transition matrix, T.

Multiple-choice questions

The following matrices are needed for Questions 1 to 7

$$U = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 43 \\ 45 \end{bmatrix} \quad W = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

1 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

2 $\det(U) =$

A 1 **B** 0 **C** 1 **D** 2 **E** 4

3 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

E not defined

4 $U^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **D** $\frac{1}{8}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

E not defined

5 $3U^2 =$

A $\begin{bmatrix} 8 & 0 \\ 7 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 12 & 0 \\ 9 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 24 & 0 \\ 21 & 3 \end{bmatrix}$

E not defined

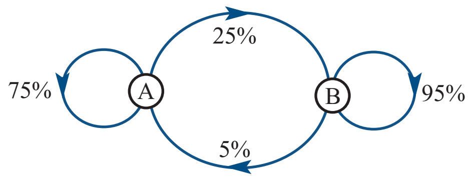
6 The matrix that could be a transition matrix is:

A U **B** V **C** W **D** X **E** Y

7 The matrix that could be a state matrix with two states is:

A U **B** V **C** W **D** X **E** Y

8



The transition matrix that can be used to represent the information in the diagram above is:

A $\begin{array}{ccc} & \text{From} & \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \text{To:} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.05 & 0.95 \end{bmatrix} & \end{array}$

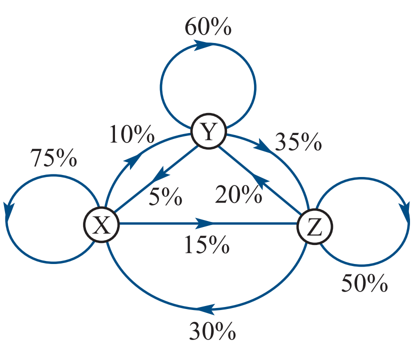
B $\begin{array}{ccc} & \text{From} & \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \text{To:} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix} & \end{array}$

C $\begin{array}{ccc} & \text{From} & \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \text{To:} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.75 & 0.95 \\ 0.25 & 0.05 \end{bmatrix} & \end{array}$

D $\begin{array}{ccc} & \text{From} & \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \text{To:} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.75 & 0.25 \\ 0.95 & 0.05 \end{bmatrix} & \end{array}$

E $\begin{array}{ccc} & \text{From} & \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} & \\ \text{To:} \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{bmatrix} 0.25 & 0.05 \\ 0.75 & 0.95 \end{bmatrix} & \end{array}$

9



The transition matrix that can be used to represent the information in the diagram above is:

A $\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} \text{X} & \text{Y} & \text{Z} \end{array} \\ \text{To:} \begin{array}{c} \text{X} \\ \text{Y} \\ \text{Z} \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.30 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$

B $\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} \text{X} & \text{Y} & \text{Z} \end{array} \\ \text{To:} \begin{array}{c} \text{X} \\ \text{Y} \\ \text{Z} \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$

C $\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To: } \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.10 & 0.15 \\ 0.60 & 0.05 & 0.35 \\ 0.50 & 0.30 & 0.20 \end{bmatrix} \end{array}$

D $\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To: } \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.10 & 0.60 & 0.20 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$

E $\begin{array}{cc} & \text{From} \\ & \begin{array}{ccc} X & Y & Z \end{array} \\ \text{To: } \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.35 & 0.50 \\ 0.10 & 0.60 & 0.20 \end{bmatrix} \end{array}$

10 Which of the following systems of linear equations has a unique solution?

I $\begin{aligned} x - 3y &= 6 \\ 2x + y &= 3 \end{aligned}$ **II** $\begin{aligned} 2x + 2y &= 6 \\ 4x + 4y &= 3 \end{aligned}$ **III** $\begin{aligned} 4x - 3y &= 6 \\ 8x - 12y &= 3 \end{aligned}$

A I only **B** I and II only **C** II only **D** I and III only **E** all

11 The linear equations $\begin{aligned} 2x - 3y &= 6 \\ 2x + y &= 3 \end{aligned}$ can be written in matrix form as:

A $\begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

C $\begin{bmatrix} 2 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

E $\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

The following information is needed for Questions 12 to 17

A system is defined by a transition matrix $T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ with $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$.

12 For this system, $S_1 =$

A $\begin{bmatrix} 60 \\ 200 \end{bmatrix}$ **B** $\begin{bmatrix} 140 \\ 160 \end{bmatrix}$ **C** $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **D** $\begin{bmatrix} 166 \\ 144 \end{bmatrix}$ **E** $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$

13 For this system, T^2 is:

A $\begin{bmatrix} 0.36 & 0.25 \\ 0.16 & 0.25 \end{bmatrix}$ **B** $\begin{bmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{bmatrix}$ **C** $\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$ **D** $\begin{bmatrix} 1.2 & 1.0 \\ 0.8 & 1.0 \end{bmatrix}$

E not defined

14 For this system, S_3 is closest to:

A $\begin{bmatrix} 160 \\ 140 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 640 \\ 560 \end{bmatrix}$ **E** $\begin{bmatrix} 400 \\ 800 \end{bmatrix}$

15 For this system, the steady state solution is closest to:

A $\begin{bmatrix} 166.5 \\ 133.5 \end{bmatrix}$ **B** $\begin{bmatrix} 166.6 \\ 133.4 \end{bmatrix}$ **C** $\begin{bmatrix} 166.7 \\ 133.3 \end{bmatrix}$ **D** $\begin{bmatrix} 166.8 \\ 133.2 \end{bmatrix}$ **E** $\begin{bmatrix} 166.9 \\ 133.1 \end{bmatrix}$

16 If $L_1 = TS_0 + B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then L_1 equals:

A $\begin{bmatrix} 70 \\ 220 \end{bmatrix}$ **B** $\begin{bmatrix} 150 \\ 180 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **D** $\begin{bmatrix} 176 \\ 164 \end{bmatrix}$ **E** $\begin{bmatrix} 210 \\ 120 \end{bmatrix}$

17 If $P_1 = TS_0 - 2\,B$, where $B = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then P_1 equals:

A $\begin{bmatrix} 140 \\ 100 \end{bmatrix}$ **B** $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 100 \end{bmatrix}$ **D** $\begin{bmatrix} 170 \\ 160 \end{bmatrix}$ **E** $\begin{bmatrix} 180 \\ 180 \end{bmatrix}$

The following information is needed for questions 18 to 19

A system of state matrices S_n is defined by the matrix equation $S_{n+1} = GS_n$, where $G = \begin{bmatrix} 0 & -0.5 \\ 1.5 & 0.5 \end{bmatrix}$

18 If $S_1 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then S_2 equals:

A $\begin{bmatrix} -12.5 \\ -2.5 \end{bmatrix}$ **B** $\begin{bmatrix} -10 \\ 25 \end{bmatrix}$ **C** $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ **D** $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$ **E** $\begin{bmatrix} 15 \\ 30 \end{bmatrix}$

19 If $S_1 = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, then S_3 equals:

A $\begin{bmatrix} -12.5 \\ -2.5 \end{bmatrix}$ **B** $\begin{bmatrix} -10 \\ 25 \end{bmatrix}$ **C** $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ **D** $\begin{bmatrix} 10 \\ 25 \end{bmatrix}$ **E** $\begin{bmatrix} 15 \\ 30 \end{bmatrix}$

20 $T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}$ is a transition matrix. $S_5 = \begin{bmatrix} 22 \\ 18 \end{bmatrix}$ is a state matrix.

If $S_5 = TS_4$, then S_4 equals:

A $\begin{bmatrix} 18 \\ 22 \end{bmatrix}$ **B** $\begin{bmatrix} 20 \\ 20 \end{bmatrix}$ **C** $\begin{bmatrix} 21.8 \\ 18.2 \end{bmatrix}$ **D** $\begin{bmatrix} 22 \\ 18 \end{bmatrix}$ **E** $\begin{bmatrix} 18.2 \\ 21.2 \end{bmatrix}$

Extended-response questions

1 We wish to solve the following system of linear equations

$$3x + 2y = 7$$
$$4x - 2y = 0$$

using matrix methods.

a Write the equations in matrix form.

b The solution is given by $X = A^{-1}C$. Write down the matrices A, A^{-1}, X and C.

c Solve the equations.

d Use the determinant test to show that the following systems of linear equations do not have a unique solution.

i $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 2 \\ 8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$

iii $\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

2 We wish to solve the following system of linear equations

$$x - 2y + z = 0$$
$$3x + 2y - z = 4$$
$$2x - y + z = 3$$

using matrix methods.

a Write the equations in matrix form.

b The solution is given by $X = A^{-1}C$. Write down the matrices A, A^{-1}, X and C.

c Solve the equations.

3 For the transition matrix $T = \begin{bmatrix} 0.15 & 0.75 \\ 0.85 & 0.25 \end{bmatrix}$ and initial state matrix $S_0 = \begin{bmatrix} 400 \\ 800 \end{bmatrix}$:

a Use the relationship $S_n = TS_{n-1}$ to determine: **i** S_1 **ii** S_2 **iii** S_3

b Determine the value of T^4.

c Use the relationship $S_n = T^n S_0$ to determine: **i** S_2 **ii** S_3 **iii** S_6

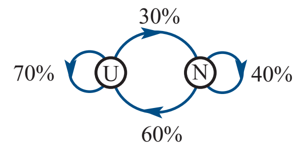
d Calculate $S_n = T^n S_0$ for $n = 10, 15, 16$ and 17 to show that the steady state solution is close to $\begin{bmatrix} 562.5 \\ 637.5 \end{bmatrix}$.

4 Experience at a fitness centre shows that:

- 70% of members who use the centre in a given week will also use the centre the next week; the remaining 30% will not use the centre the next week
- 40% of members who do not use the centre in a given week will use the centre in the next week; the remaining 60% will not use the centre in the next week.

a Construct a transition matrix, T, of the form below to describe this situation.

$$\begin{array}{r l c}
 & & \text{This week} \\
 & & \begin{array}{cc} \textit{Use} & \textit{Not use} \end{array} \\
\text{Next week:} & \begin{array}{r} \textit{Use} \\ \textit{Not use} \end{array} & \left[\begin{array}{cc} \quad & \quad \\ \quad & \quad \end{array}\right]
\end{array}$$



b This week, 400 members used the centre and 100 did not use the centre. Write down a column matrix, S_0, that describes this situation.

c How many members do you expect to use the centre next week?

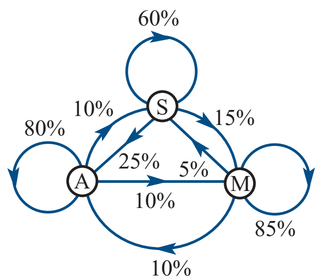
d How many members do you expect to use the centre after five weeks?

e In the long term, how many members are expected to use the centre each week?

5 Camper vans are available for weekly rental in each of three cities: Sydney, Melbourne and Adelaide. At the beginning of the year, there are 120 camper vans available for hire in Sydney (S), 200 in Melbourne (M) and 80 in Adelaide (A).

a Write down a column matrix, S_0, that lists the number of vans initially available for hire in the three cities.

The transition diagram opposite shows the percentage of vans hired in each city and either returned to that city or to one of the other cities at the end of each week.



b Write down the percentage of camper vans hired in Melbourne that will be returned to Adelaide the next week.

c Two hundred vans are rented in Melbourne in the first week. How many of these vans will be returned to either Adelaide or Sydney the next week?

d Use the transition diagram to construct a transition matrix, T.

Assume that all camper vans available for hire are rented each week and that all camper vans returned to a city are available for hire the next week.

e Write down an expression in terms of S_0 and T that can be used to determine the state matrix, S_1, that lists the number of camper vans available for hire in the three cities at the end of week 1. Determine this state matrix.

f Use the transition matrix to predict the number of camper vans available for hire in Sydney at the end of week 3. Give your answer correct to the nearest whole number.

g In the long term, how many vans are expected to be available for hire at the end of each week in the three cities? Give your answers correct to the nearest whole numbers.

6 The Dinosaurs (D) and the Scorpions (S) are two basketball terms that play in different leagues in the same city.

The matrix A_1 is the attendance matrix for the first game. This matrix shows the number of people who attended the first Dinosaurs game and the number of people who attended the first Scorpions game.

$$A_1 = \begin{bmatrix} 2000 \\ 1000 \end{bmatrix} \begin{matrix} D \\ S \end{matrix}$$

The number of people expected to attend the second game for each team can be determined using the matrix equation

$$A_2 = GA_1$$

where G is the matrix

$$G = \begin{array}{c c} & \textit{this game} \\ & \begin{matrix} D & \quad S \end{matrix} \\ & \begin{bmatrix} 1.2 & -0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{matrix} D \\ S \end{matrix} \textit{ next game} \end{array}$$

a **i** Determine A_2, the attendance matrix for the second game.

ii Every person who attends either the second Dinosaurs game or the second Scorpions game will be given a free cap.
How many caps, in total, are expected to be given away?

Assume that the attendance matrices for successive games can be determined as follows.

$A_3 = GA_2,\ \ A_4 = GA_3$, and so on such that $A_{n+1} = GA_n$

b Determine the attendence matrix (with the elements written correct to the nearest whole number) for game 10.

c Describe the way in which the number of people attending the Dinosaurs' games is expected to change over the next 80 or so games.

The attendance at the first Dinosaurs game was 2000 people and the attendance at the first Scorpions game was 1000 people.

Suppose, instead, that 2000 people attend the first Dinosaurs game, and 1800 people attend the first Scorpions game.

d Describe the way in which the number of people attending the Dinosaurs' games is expected to change over the next 80 or so games. [VCAA 2010]

8 In the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 3 \\ -5 & -4 & 7 \end{bmatrix}$ the element $a_{3,2} = a_{32} =$

A -4 **B** -1 **C** 0 **D** 3 **E** 4

9 $3\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} =$

A $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & 0 \\ -4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 8 & 0 \\ -5 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$

10 $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$

A $[-2]$ **B** -2 **C** $[4]$ **D** 4 **E** not defined

11 R is a 3×3 matrix. The order of matrix R^4 is:

A 3 **B** 3×3 **C** 4×4 **D** 9 **E** 9×9

12 $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix} =$

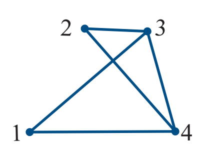
A $\begin{bmatrix} -2 & 20 \\ 0 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 8 \\ 0 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 4 & -1 & 2 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} -2 & 0 & 6 \\ 2 & -1 & 2 \\ 3 & 0 & 7 \end{bmatrix}$ **E** not defined

13 Matrix X is of order $p \times q$ and matrix Y is of order $q \times r$.
The matrix products $X^{-1}Y$ and $X^{-1}Y$ are both defined:

A for no values of p, q or r **B** when $p = r$ **C** when $p = q = r$ only
D when $p = q$ and $q = r$ **E** for all values of p, q and r

14 The diagram opposite is to be represented by a matrix A, where:

- element $= 1$ if the two points are joined by a line
- element $= 0$ if the two points are not connected



The matrix A is:

A $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

15 The matrix equation $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ generates the following pair of simultaneous linear equations:

A $\begin{aligned} x + y &= 1 \\ x + 3y &= 4 \end{aligned}$ **B** $\begin{aligned} 2x &= 1 \\ x + 3y &= 4 \end{aligned}$ **C** $\begin{aligned} x + 4y &= 0 \\ x + 3y &= 4 \end{aligned}$ **D** $\begin{aligned} x &= 1 \\ 3x + y &= 4 \end{aligned}$ **E** $\begin{aligned} x + y &= 1 \\ 3x + y &= 4 \end{aligned}$

The following matrices are needed for Questions 16 to 23

$$U = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix} \quad V = \begin{bmatrix} 0.75 & 0.35 \\ 0.15 & 0.45 \end{bmatrix} \quad W = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

16 The matrix that cannot be raised to a power is:

A U **B** V **C** W **D** X **E** Y

17 $\det(X) =$

A -1 **B** -0.25 **C** 0.25 **D** 1 **E** 0.75

18 $Y^{-1} =$

A $\begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -1 \\ -0.5 & 0.5 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix}$ **D** $\frac{1}{4}\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ **E** not defined

19 Which of the following matrix expressions is not defined?

A $VX - XV$ **B** $UW - W$ **C** WZ **D** $Y^2 - VX^{-1}$ **E** $YW + WX$

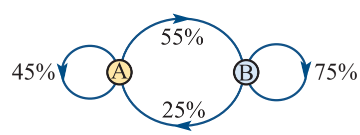
20 $(U - Y)^2 =$

A $\begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 21 \\ 21 & 45 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 7 \\ 5 & 14 \end{bmatrix}$ **D** $\begin{bmatrix} 44 & 119 \\ 85 & 231 \end{bmatrix}$ **E** not defined

21 The matrix that could be a transition matrix is:

A U **B** V **C** W **D** X **E** Y

22 The transition matrix that can be used to represent the information in the diagram opposite is:



A From; To:
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix} \end{array}$$

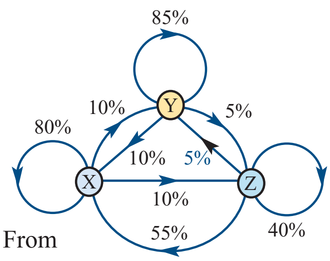
B From; To:
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.45 & 0.25 \\ 0.55 & 0.75 \end{bmatrix} \end{array}$$

C From; To:
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.45 & 0.75 \\ 0.35 & 0.25 \end{bmatrix} \end{array}$$

D From; To:
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.75 & 0.45 \\ 0.55 & 0.25 \end{bmatrix} \end{array}$$

E From; To:
$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.55 & 0.25 \\ 0.45 & 0.75 \end{bmatrix} \end{array}$$

23 The transition matrix that can be used to represent the information in the diagram below is:



A $\begin{array}{c} \\ \\ \text{To: } X \\ Y \\ Z \end{array} \begin{array}{c} \text{From} \\ \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.80 & 0.85 & 0.40 \\ 0.10 & 0.10 & 0.05 \\ 0.10 & 0.05 & 0.55 \end{bmatrix} \end{array}$

B $\begin{array}{c} \\ \\ \text{To: } X \\ Y \\ Z \end{array} \begin{array}{c} \text{From} \\ \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.80 & 0.10 & 0.55 \\ 0.10 & 0.85 & 0.5 \\ 0.10 & 0.5 & 0.40 \end{bmatrix} \end{array}$

C $\begin{array}{c} \\ \\ \text{To: } X \\ Y \\ Z \end{array} \begin{array}{c} \text{From} \\ \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.80 & 0.10 & 0.55 \\ 0.10 & 0.85 & 0.05 \\ 0.10 & 0.05 & 0.40 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ \\ \text{To: } X \\ Y \\ Z \end{array} \begin{array}{c} \text{From} \\ \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.10 & 0.85 & 0.55 \\ 0.80 & 0.15 & 0.5 \\ 0.10 & 0.5 & 0.40 \end{bmatrix} \end{array}$

E $\begin{array}{c} \\ \\ \text{To: } X \\ Y \\ Z \end{array} \begin{array}{c} \text{From} \\ \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.80 & 0.10 & 0.55 \\ 0.85 & 0.85 & 0.05 \\ 0.40 & 0.05 & 0.40 \end{bmatrix} \end{array}$

24 Which of the following systems of linear equations does *not* have a unique solution?

I $x - 3y = 5$, $2x + 2y = 3$ **II** $2x + y = 6$, $5x + 4y = 3$ **III** $2x - 3y = 2$, $4x - 6y = 1$

A I only **B** I and II only **C** II only **D** III only **E** all

25 The solution of the matrix equation $\begin{bmatrix} 11 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is given by $\begin{bmatrix} x \\ y \end{bmatrix} =$

A $\begin{bmatrix} 11 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 5 \\ 2 & 11 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **C** $\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 11 \end{bmatrix}$

D $\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 11 & -5 \\ -2 & 1 \end{bmatrix}$ **E** $\dfrac{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}{\begin{bmatrix} 11 & -5 \\ -2 & 1 \end{bmatrix}}$

26 $T = \begin{bmatrix} 0.95 & 0.23 \\ & 0.77 \end{bmatrix}$ is a transition matrix. One element is missing. The missing element is:

A 0.05 **B** 0.5 **C** 0.18 **D** 0.33 **E** 5

32 Kerry sat for a multiple-choice test consisting of six questions.
Each question had four alternative answers, *A, B, C* or *D*.
He selected *D* for his answer to the first question.
He then determined the answers to the remaining questions by following the transition matrix

$$\text{Next question}\quad \begin{array}{c} \\ A\\ B\\ C\\ D \end{array}\!\!\begin{array}{c} \textit{This question} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

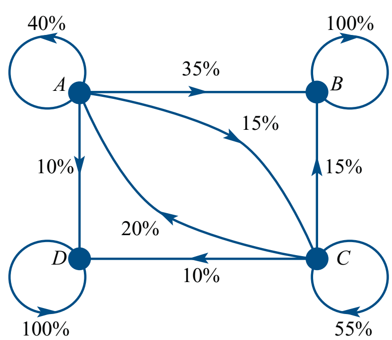
The answers that he gave to the six test questions, starting with *D*, were:

A *DBCADB* **B** *DBCAAA* **C** *DBCACA* **D** *DACBDD* **E** *DCBABC*

[VCAA 2007]

The following information relates to questions 33 to 35.
A large population of mutton birds migrates each year to a remote island to nest and breed.
There are four nesting sites on the island, *A*, *B*, *C* and *D*.
Researchers suggest that the following transition matrix can be used to predict the number of mutton birds nesting at each of the four sites in subsequent years. An equivalent transition diagram is also given.

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0.2 & 0\\ 0.35 & 1 & 0.15 & 0\\ 0.15 & 0 & 0.55 & 0\\ 0.1 & 0 & 0.1 & 1 \end{bmatrix} \end{array}\!\!\begin{array}{c} \\ \\ A\\ B\\ C\\ D \end{array}\quad \textit{next year}$$



33 Two thousand eight hundred mutton birds nest at site *C* in 2008.
Of these 2800 mutton birds, the number that nest at site *A* in 2009 is predicted to be:

A 560 **B** 980 **C** 1680 **D** 2800 **E** 3360

34 This transition matrix predicts that, in the long term, the mutton birds will:

A nest only at site *A*
B nest only at site *B*
C nest only at sites *A* and *C*
D nest only at sites *B* and *D*
E continue to nest at all four sites

3 We wish to solve the following system of linear equations

$$\begin{aligned} x - y + z &= 2 \\ 3x + y - z &= 2 \\ x - y \quad\; &= 5 \end{aligned}$$

using matrix methods.

a Write the equations in matrix form.

b The solution is given by $X = A^{-1}C$. Write down the matrices A, A^{-1}, X and C.

c Solve the equations.

4 Lake Blue and Lake Green are two small lakes connected by a channel. This enables fish to move between the two lakes on a daily basis. Research has shown that each day:

- 67% of fish in Lake Blue stay in Lake Blue
- 33% of fish in Lake Blue move to Lake Green
- 72% of fish in Lake Green stay in Lake Green
- 28% of fish in Lake Green move to Lake Blue.

a Construct a transition matrix, T, of the form:

$$\begin{array}{ll} & \text{From} \\ & \begin{array}{cc} \textit{Blue} & \textit{Green} \end{array} \\ \text{To:} \begin{array}{r} \textit{Blue} \\ \textit{Green} \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

to describe this situation.

b Today there are currently 4000 fish in Lake Blue and 6000 fish in Lake Green. Write down a column matrix, S_0, that describes this situation.

c How many fish do you expect to be in each lake tomorrow?

d How many fish do you expect to be in each lake in three days' time?

e In the long term, how many fish do you expect to be in each lake?

5 For the transition matrix $T = \begin{bmatrix} 0.86 & 0.2 \\ 0.14 & 0.8 \end{bmatrix}$ and an initial state matrix $S_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$:

a Use the relationship $S_n = TS_{n-1}$ to determine: **i** S_1 **ii** S_2 **iii** S_3

b Determine the value of T^6.

c Use the relationship $S_n = T^n S_{n-1}$ to determine: **i** S_2 **ii** S_3 **iii** S_5

d Calculate $S_n = T^n S_0$ for $n = 10$, 15, 20 and 21 to show that the steady state solution is close to $\begin{bmatrix} 176.5 \\ 123.5 \end{bmatrix}$.

6 The table below displays the energy content and amounts of fat, carbohydrate and protein contained in a serve of four foods: bread, margarine, peanut butter and honey.

Food	**Energy content (kilojoules/serve)**	**Fat (grams/serve)**	**Carbohydrate (grams/serve)**	**Protein (grams/serve)**
Bread	531	1.2	20.1	4.2
Margarine	41	6.7	0.4	0.6
Peanut butter	534	10.7	3.5	4.6
Honey	212	0	12.5	0.1

a Write down a 2×3 matrix that displays the fat, carbohydrate and protein content (in columns) of bread and margarine.

b A and B are two matrices defined as follows.

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 531 \\ 41 \\ 534 \\ 212 \end{bmatrix}$$

i Evaluate the matrix product AB. **ii** Determine the order of matrix product BA.

Matrix A displays the number of servings of the four foods (bread, margarine, peanut butter and honey) needed to make a peanut butter and honey sandwich.

Matrix B displays the energy content per serving of the four foods.

iii Explain the information that the matrix product AB provides.

c The number of serves of bread (b), margarine (m), peanut butter (p) and honey (h) that contain, in total, 53 grams of fat, 101.5 grams of carbohydrate, 28.5 grams of protein and 3568 kilojoules of energy can be determined by solving the matrix equation:

$$\begin{bmatrix} 1.2 & 6.7 & 10.7 & 0 \\ 20.1 & 0.4 & 3.5 & 12.5 \\ 4.2 & 0.6 & 4.6 & 0.1 \\ 531 & 41 & 534 & 212 \end{bmatrix} \begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 53 \\ 101.5 \\ 28.5 \\ 3568 \end{bmatrix}$$

Solve the matrix equation to find the values b, m, p and h.

[VCAA 2007]

7 To study the life-and-death cycle of an insect population, a number of insect eggs (E), juvenile insects (J) and adult insects (A) are placed in a closed environment.

The initial state of this population can be described by the column matrix:

$$S_0 = \begin{array}{c} \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} E \\ J \\ A \\ D \end{array}$$

A row has been included in the state matrix to allow for insects and eggs that die (D).

a What is the total number of insects in the population (including eggs) at the beginning of the study?

In this population:

- eggs may die, or they may live and grow into juveniles
- juveniles may die, or they may live and grow into adults
- adults will live a period of time but they will eventually die.

In this population, the adult insects have been sterilised so that no new eggs are produced. In these circumstances, the life-and-death cycle of the insects can be modelled by the transition matrix:

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cccc} E & J & A & D \end{array} \\ T = & \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array} \text{ next week} \end{array}$$

b What proportion of eggs turn into juveniles each week?

c **i** Evaluate the matrix product $S_1 = TS_0$.

ii Write down the number of live juveniles in the population after one week.

iii Determine the number of live juveniles in the population after four weeks. Write your answer correct to the nearest whole number.

iv After a number of weeks there will be no live eggs (less than one) left in the population. When does this first occur?

v Write down the exact long-term state matrix for this population.

d If the study is repeated with unsterilised adult insects, eggs will be laid and potentially grow into adults.

Assuming 30% of adults lay eggs each week, the population matrix after one week, S_1, is now given by

$$S_1 = TS_0 + BS_0$$

where $B = \begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array}$

i Determine S_1.

This pattern continues. The population matrix after n weeks S_n, is given by

$$S_n = TS_{n-1} + BS_{n-1}$$

ii Determine the number of live eggs in this insect population after two weeks.

[VCAA 2007]

8 The following transition matrix, T, is used to help predict class attendance of History students at the university on a lecture-by-lecture basis.

$$\begin{array}{cc} & \text{this lecture} \\ & \begin{array}{cc} \textit{attend} & \textit{not attend} \end{array} \\ T = & \begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array} \quad \text{next lecture} \end{array}$$

S_1 is the attendance matrix for the first History lecture.

$$S_1 = \begin{bmatrix} 540 \\ 36 \end{bmatrix} \begin{array}{l} \textit{attend} \\ \textit{not attend} \end{array}$$

S_1 indicates that 540 History students attended the first lecture and 36 History students did not attend the first lecture.

a Use T and S_1 to:

i to the nearest whole number, determine S_2, the attendance matrix for the second lecture

ii predict the number of History students attending the fifth lecture

b Write down a matrix equation for S_n in terms of T, n and S_1.

The History lecture can be transferred to a smaller lecture theatre when the number of students predicted to attend falls below 400.

c For which lecture can this first be done?

d In the long term, how many History students are predicted to attend lectures?

The bookshop manager at the university has developed a matrix formula for determining the number of Mathematics and Physics textbooks he should order each year.

For 2009, the starting point for the formula is the column matrix S_{2008}. This lists the number of Mathematics and Physics textbooks sold in 2008.

$$S_{2008} = \begin{bmatrix} 456 \\ 350 \end{bmatrix} \begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array}$$

O_{2009} is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for 2009. O_{2009} is given by the matrix formula:

$$O_{2009} = AS_{2008} + B \qquad \text{where } A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix} \qquad \text{and } B = \begin{bmatrix} 18 \\ 12 \end{bmatrix}$$

e Determine O_{2009}.

The matrix formula above only allows the manager to predict the number of books he should order one year ahead.

A new matrix formula enables him to determine the number of books to be ordered two or more years ahead.

The new matrix formula is:

$$O_{n+1} = CO_n - D$$

where O_n is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for year n.

Here $C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$ and $D = \begin{bmatrix} 40 \\ 38 \end{bmatrix}$

The number of books ordered in 2008 was given by:

$$O_{2008} = \begin{bmatrix} 500 \\ 360 \end{bmatrix} \begin{array}{l} \textit{Mathematics} \\ \textit{Physics} \end{array}$$

f Use the new matrix formula to determine the number of Mathematics textbooks the bookshop manager should order in 2010.

[VCAA 2008]

9 In 2009, a school entered a Rock Eisteddfod competition.

When rehearsals commenced in February, all students were asked whether they thought the school would make the state finals. The students' responses, 'yes', 'no' or 'undecided' are shown in the initial state matrix S_0.

$$S_0 = \begin{bmatrix} 160 \\ 120 \\ 220 \end{bmatrix} \begin{array}{l} \textit{yes} \\ \textit{no} \\ \textit{undecided} \end{array}$$

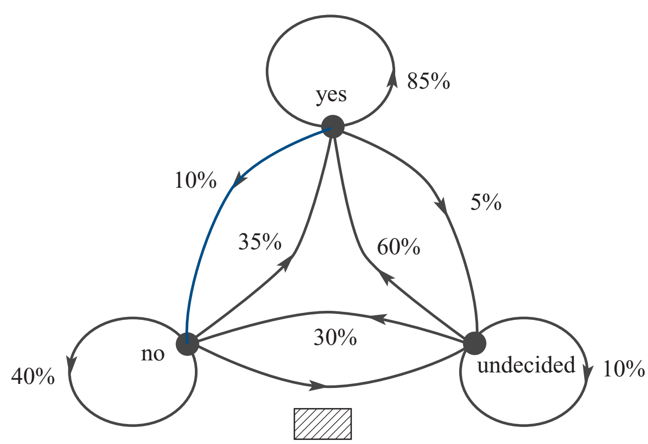
a How many students attend this school?

Each week some students are expected to change their responses. The changes in their responses from one week to the next are modelled by the transition matrix, T, shown below.

$$\begin{array}{c} \textit{response this week} \\ \begin{array}{ccc} \textit{yes} & \textit{no} & \textit{undecided} \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.85 & 0.35 & 0.60 \\ 0.10 & 0.40 & 0.30 \\ 0.05 & 0.25 & 0.10 \end{bmatrix} \begin{array}{l} \textit{yes} \\ \textit{no} \\ \textit{undecided} \end{array} \quad \begin{array}{c} \textit{response} \\ \textit{next week} \end{array}$$

The following diagram can also be used to display the information represented in the transition matrix T.



Appendix: TI-Nspire CAS

Operating system

Written for operating system OS3.

Keystroke actions and short cuts for the TI-Nspire CAS Clickpad (grey)

esc: removes menus and dialogue boxes
ctrl + esc: undo last move

caps + esc: redo last move

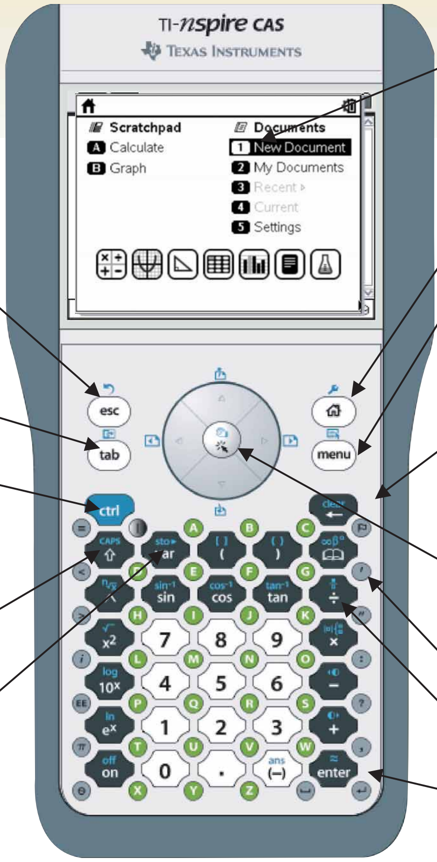
tab: move to next entry box (field)

ctrl + tab: switch applications in split screen

ctrl: accesses secondary (blue) commands
ctrl + ▲: displays page sorter
ctrl + ◀: displays previous page
ctrl + ▶: displays next page

caps: shift key

sto→ var: display lists of stored variables
ctrl + sto→ var: store command (→)



Home page (icon display)

home: displays icon page to select applications, mode, My Documents and start a new document

menu: options for each application
ctrl + menu: contextual menus (same as right mouse click)

clear: backspace – deletes a character
ctrl + clear: clear – clears an entry line

mouse pointer (cursor), selects items
ctrl + mouse pointer: grab

ctrl + degree key: degree sign (°)

ctrl + fraction key: fraction template

enter: completes commands and displays results

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Keystroke actions and short cuts for the TI-Nspire CAS Touchpad (black)

esc: removes menus and dialogue boxes
ctrl+esc: undo last move
shift+esc: redo last move

tab: move to next entry box (field)
ctrl+tab: switch applications in split screen

Nav Pad (touch pad)

ctrl: accesses secondary (yellow) commands

ctrl+▲: displays page sorter
ctrl+◀: displays previous page
ctrl+▶: displays next page

shift: shift key

var: display lists of stored variables
ctrl+var: store command (→)

Home page (icon display)

on: displays icon page to select applications, mode, My Documents and start a new document.

menu: options for each application.
ctrl+menu: contextual menus (same as right mouse click)

: mouse pointer (cursor), selects items
ctrl+ : grab

del: backspace – deletes a character.

: 2D templates

ctrl+÷: fraction template

enter: completes commands and displays results.

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Keystroke actions and short cuts for the TI-Nspire CAS CX

esc: removes menus and dialogue boxes
ctrl + esc: undo last move
shift + esc: redo last move

tab: move to next entry box (field)
ctrl + tab: switch applications in split screen

Nav Pad (touch pad)

ctrl: accesses secondary (yellow) commands

ctrl + ▲: displays page sorter
ctrl + ◀: displays previous page
ctrl + ▶: displays next page

shift: shift key

var: display lists of stored variables
ctrl + var: store command (→)

Home page (icon display)

on: displays icon page to select applications, mode, My Documents and start a new document.

menu: options for each application.
ctrl + menu: contextual menus (same as right mouse click)

: mouse pointer (cursor), selects items
ctrl + : grab

del: backspace – deletes a character.

: 2D templates

ctrl + ÷: fraction template

enter: completes commands and displays results.

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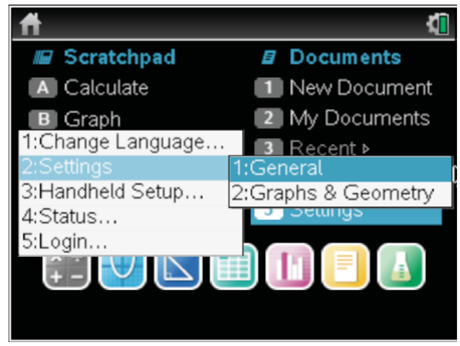
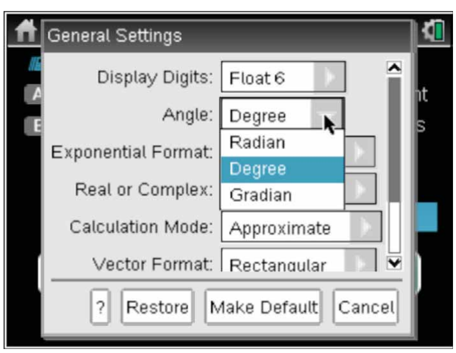
Setting the Mode

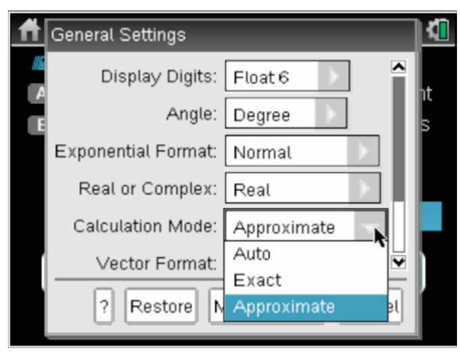
For Further Mathematics it is convenient to set the calculation mode to **Degree** and **Approximate** (decimal) right from the start. The calculator will remain in this mode unless you change the settings again.

1 Press ⌂on (or off/on and ⌂ on the grey Clickpad) and select **Settings>Settings>General** as shown in step **2** below.
Hint: If the icon page does not display when you press ⌂on, then press esc first.

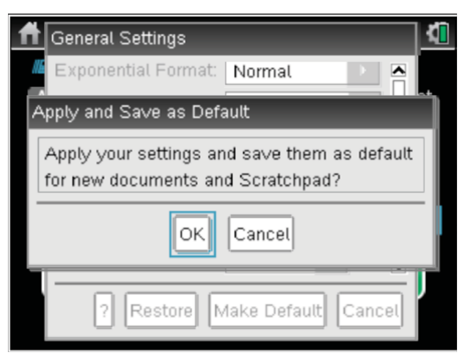


2 Use the tab key to move down to the box that displays **Angle**. Use ► ▼ to select **Degree**, press enter. Continue using the tab key until you reach the **Calculation Mode** box. Use ► ▼ to select **Approximate**, then press enter.





3 Press the tab key until you reach the **Make Default** box and press enter. Press enter again to accept the change to the settings.



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The home screen is divided into two main areas – **Scratchpad** and **Documents**.

Scratchpad

Note: The Scratchpad is only available on OS 2.0 or higher.

A: Calculate – this is a fully functional CAS calculation platform that allows for quick and easy access to the home screen and menus. It can be used for most calculations such as arithmetic, algebra, finance, trigonometry and matrices. Scatchpad is similar in functionality to the **Documents: Calculator** application, but saves opening up a new document every time you want to do a calculation.

Calculating

1 Press then (home on) (or (off on) and (home) on the grey Clickpad).
Pressing (scratchpad) also opens the **Scratchpad**.
If you prefer to use the **Documents** platform for your calculations then press (home)**>New Document>Add Calculator** and follow the same steps.
Hint: You can undo your action using (ctrl) + (esc) (or (ctrl) + (Z))
Note: If using the TI-Nspire CAS Touchpad you can check the mode settings by moving the cursor onto the icon displayed on the top of each screen. (General settings only.)

2 To calculate, simply enter the required expression and press (enter). For example, if we wish to evaluate $5 + 3 \times 6$, we type in the expression and press (enter).
Note: The symbol · (dot) is used on the screen to represent the multiplication sign.

3 Type in $56 \div 9$ and press (enter) to obtain the result 6.22222. If the result is displayed as $\frac{56}{9}$, you are in **Auto** not **Approximate** mode. Change the mode now (see above).

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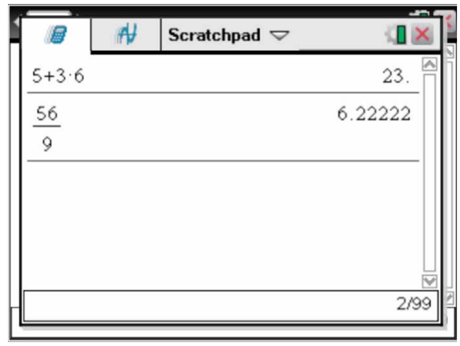
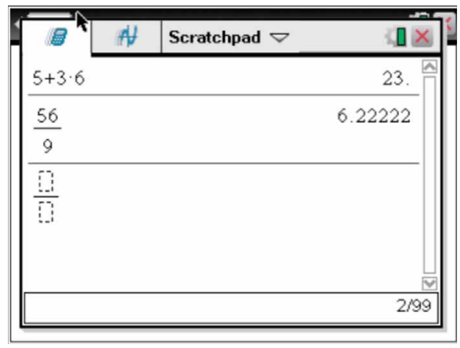
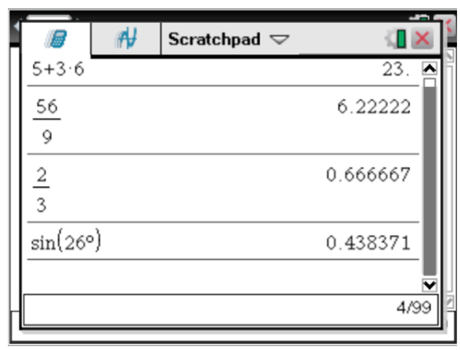
4 To copy and edit a previously evaluated line, keep pressing ▲ until the expression to be copied is highlighted, then press (enter) to paste it on a new line. You are now able to edit this expression as required.
Note: You can only edit on a new line; that is, if no answer is present on that line.



5 You can enter fractions using the fraction template if you prefer. Press (ctrl) + (÷) to paste the fraction template and enter the values. Use the (tab) key to move between boxes. Press (enter) to display the answer.



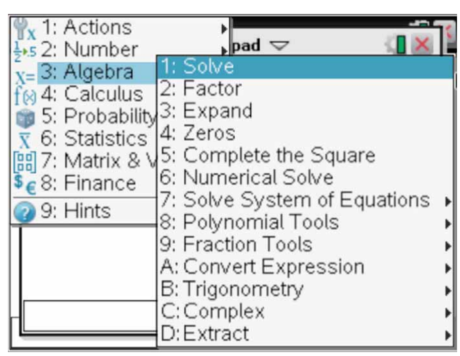
6 For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.
Note: if the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.
The degree symbol can be accessed using (?!▸) (or (ctrl) + (ʹ) on the Clickpad). Alternatively select from the Symbols palette ((ctrl) + (📖)).



Solving equations

Using the **Solve command**

Solve $2y + 3 = 7$ for y.



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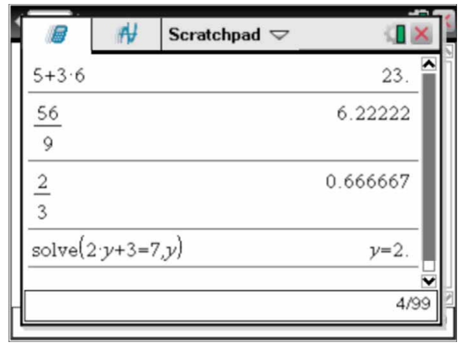
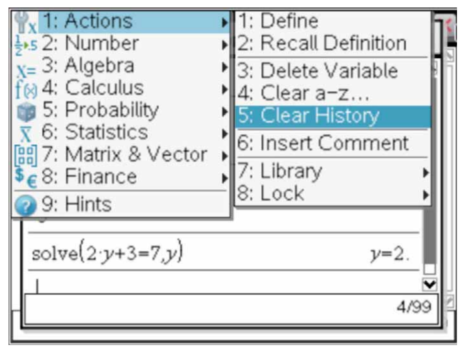
1 Go to the **Scratchpad: Calculate** and press (menu)**>Algebra>Solve** and complete as shown opposite.
Keystrokes: (2) [Y] (+) (3) (=) (7) (,) [Y] (enter).
Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.
Note that the **Scratchpad: Calculate** continues on from the previous work. Use the up arrow ▲ to scroll back through previously worked examples.

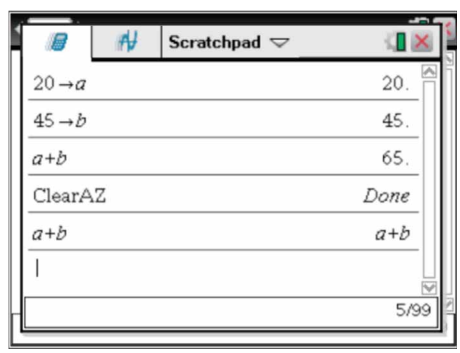


Clearing the history area

Once you have pressed (enter) the computation becomes part of the History area.
To clear a line from the history area, press ▲ repeatedly until the expression is highlighted and press (del) (or press (clear) on the Clickpad).
To completely clear the History Area, press (menu)**>Actions>Clear History**.



Alternatively, press (ctrl)+(menu) to access the contextual menu.
It is also useful to occasionally clear any previously stored values. It is important you do this before a test or examination. Clearing History does not clear stored variables.
Pressing (menu)**>Actions>Clear a-z. . .** will clear any stored values for single letter variables that have been used in the Scratchpad.
Use (menu)**>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable *perimeter*, then use **DelVar** *perimeter*.



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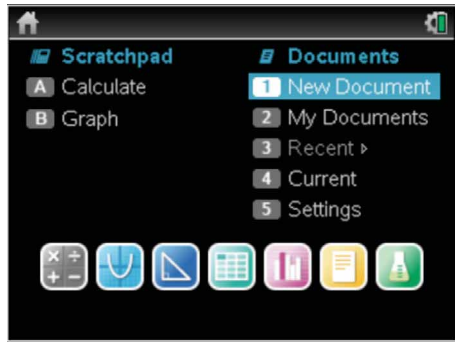
Documents

This platform must be used to cover most parts of the **Core** section, particularly statistical plotting and spreadsheet applications. Also many of the finance applications from the **Business-related mathematics** module such as depreciation and sections of the **Number patterns and applications** module will be covered using this platform.

All of the examples described in the **Scratchpad** section earlier can also be done using the **Documents** platform if preferred.



Starting a new document

1 To start a new document, press (⌂ on) (or press (⌂) on the Clickpad) and select **New Doc** (or use the keyboard shortcut (ctrl) + [N] to start a new document).

2 If prompted to save an existing document move the cursor to **No** and press (enter).
It is possible to switch between the **Scratchpad** and the **Documents** platforms by pressing (⌂ on) or (▤). This might be useful to do a quick trial, or check, a calculation or graph in the **Scratchpad** whilst working in the **Documents** platform.
Select **Current Document** on the **Home** page to return to the current document.

How to construct parallel boxplots

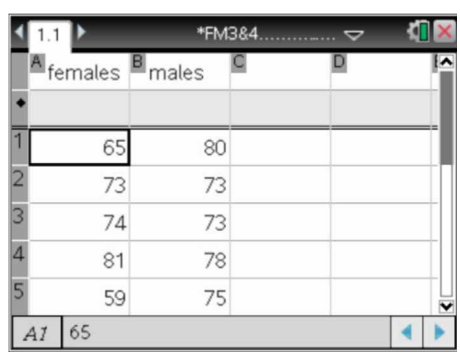
Construct parallel boxplots to display the pulse rates of 23 adult females and 23 adult males.

Pulse rate (beats per minute)	
Females	Males
65 73 74 81 59 64 76 83 95 70 73 79	80 73 73 78 75 65 69 70 70 78 58 77
64 77 80 82 77 87 66 89 68 78 74	64 76 67 69 72 71 68 72 67 77 73

Steps

1 Start a new document: (ctrl) + [N].

2 Select **Add Lists & Spreadsheet**.
Enter the data into lists called ***females*** and ***males*** as shown.



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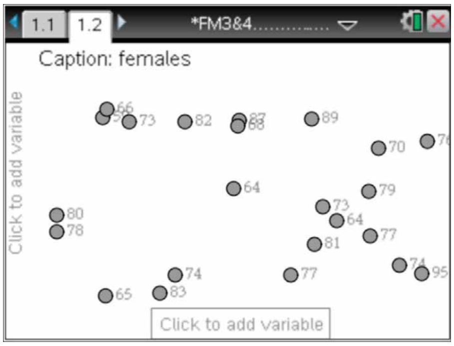
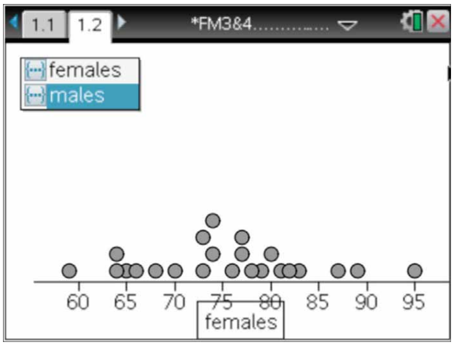
Cambridge University Press

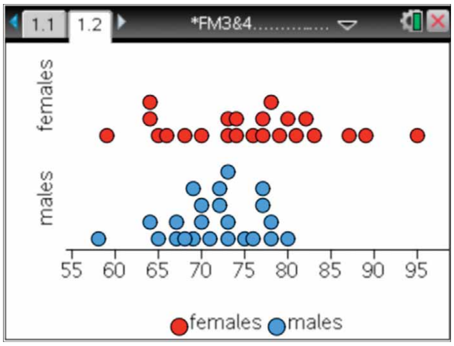
3 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + I and select **Add Data & Statistics**.
(or press on and arrow to [icon] and press enter)
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



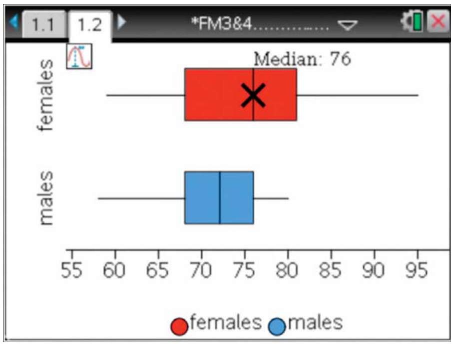
a Press tab to show the list of variables. Select the variable, *females*. Press enter to paste the variable to the *x*-axis. A dot plot is displayed by default, as shown.



b To add another variable to the *x*-axis, press menu **>Plot Properties>Add X Variable**, then enter. Select the variable ***males***. Parallel dot plots are displayed by default.



c To change the plots to box plots press menu **>Plot Type>Box Plot**, then enter.
Note: To change the colour of the plots, move the cursor on the plot and press ctrl + menu **>Color>Fill Color**.
Your screen should now look like that shown opposite.



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4 Data analysis

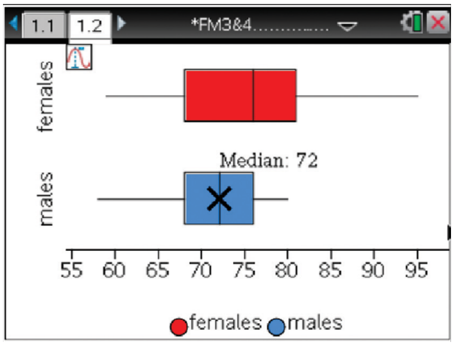
Use (menu) **>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points.

Starting at the far left of the plots, we see that, for females, the:

- minimum value is 59: **MinX = 59**
- first quartile is 68: **Q1 = 68**
- median is 76: **Median = 76**
- third quartile is 81: **Q3 = 81**
- maximum value is 95: **MaxX = 95**

and for males, the:

- minimum value is 58: **MinX = 58**
- first quartile is 68: **Q1 = 68**
- median is 72: **Median = 72**
- third quartile is 76: **Q3 = 76**
- maximum value is 80: **MaxX = 80**



ISBN 978-1-107-65590-4

Cambridge University Press